# Robust Synchronization of Interconnected Linear Systems over Intermittent Communication Networks

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***Abstract*— The property of synchronization of multiple linear time-invariant systems connected over a network with stochastically-driven isolated communication events is studied. We propose a solution to the problem of designing a feedback controller that, using information obtained over such networks, asymptotically drives the values of their states to synchronization and renders such a condition Lyapunov stable. To solve this problem, we propose a controller with hybrid dynamics, namely, the controller exhibits continuous dynamics between communication events and, at such events, has variables that jump. Due to the additional continuous and discrete dynamics inherent to the networked systems and communication structure, we use a hybrid systems framework to model the closed-loop system and design the controller. The problem of synchronization is then recast as a compact set stabilization problem and, by employing Lyapunov stability tools for hybrid systems, sufficient conditions for asymptotic stability of the synchronization set are provided. Furthermore, we show that the synchronization property is robust to a class of perturbations on the transmitted data. Numerical examples illustrating the main results are included.**

## I. Introduction

Synchronization in networked systems has been extensively studied in the literature and has been approached by various viewpoints and methodologies. Specifically, synchronization of coupled identical linear systems has been throughly investigated in both continuous and discrete time domains [1], [2]. Further studies involving nonlinear systems within complex network structures are also available [3]. A typical approach is to study the network structure using graph theory, which provides a solid understanding of the connectivity of the network and its effect on the individual dynamics of the systems [4], [5]. The study of the stability of synchronization using systems theory tools, like Lyapunov functions [6], [7], contraction theory [8], and incremental input-to-state stability [9], [10] has also been explored. Notably, recent research efforts on sample-data systems [11] and event-triggered control [12] for the stabilization of sets provide results that can become useful for synchronization. Furthermore, synchronization of nonlinear systems with both continuous and impulsive update laws has been studied in [13], [14].

This article deals with the problem of synchronization of the states between multiple continuous linear time-invariant (LTI) systems under general connected directed graphs when information for the agents is only available at stochastically determined instants. To solve this problem, we design a dynamic hybrid state-feedback controller that continuously evolves between communication times and undergoes an instantaneous change in its state when a new measurement is available. Due to the combination of continuous and impulsive dynamics, we use hybrid systems theory to model the systems, the controllers, and network topologies, and apply Lyapunov stability tools. More precisely, we recast the synchronization problem as the stabilization of a compact set and apply a Lyapunov theorem for hybrid systems. Our main contributions contain the mathematical modeling of such networked systems, controller design, and sufficient conditions for synchronization of multiple interconnected LTI systems via a directed graph. We consider the stochastically-driven intermittent network over two cases of communication graph structures: a strongly connected directed graph and a completely connected directed graph. Furthermore, we show that the algorithm induces robustness to a class of perturbations on the transmitted data.

The remainder of this paper is organized as follows. In Section II, we introduce the impulsive network model, control structure, and recast the closed loop system into a hybrid systems framework. In Section III, we present the models, results and give examples. Finally, in Section III-C we show that the stability of the set capturing synchronization is robust to a class of perturbations.

**Basic Notation:** The set $\mathbb{R}$ denotes the space of real numbers. The set $\mathbb{R}^n$ denotes the $n$-dimensional Euclidean space. The set $\mathbb{N}$ denotes the natural numbers including zero, i.e., $\mathbb{N} = \{0, 1, 2, ...\}$. Given a set $S \subset \mathbb{R}^n$, $S^N$ is defined as $S \times S \times \cdots \times S$, namely, the cartesian product of $S$ with itself $N$ times. Given two vectors $x, y$, we denote $(x, y) = [x^\top, y^\top]^\top$. Given $x \in \mathbb{R}^n$, $|x|$ denotes the Euclidean norm of $x$. The set $\mathbb{B}$ is the closed unit ball centered around the origin in Euclidean space. The identity matrix is denoted by $I$. For a matrix $A \in \mathbb{R}^{n\times m}$, $A^\top$ denotes the transpose of $A$ and $|A|$ denotes the induced 2-norm. For two symmetric matrices with same dimensions, $A$ and $B$, $A > B$ means that $A - B$ is positive definite. Given two matrices $A = \{a_{ij}\} \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{p\times q}$, the Kronecker product between $A$ and $B$ is defined as $A \otimes B = \begin{bmatrix} a_{11}B & \dots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \dots & a_{mn}B \end{bmatrix}$. A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to belong to class $\mathcal{K}$ if is continuous, zero at zero, and strictly increasing; it is a class $\mathcal{K}_\infty$ function if it belongs to class $\mathcal{K}$ and is unbounded. Given a closed set $\mathcal{A} \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$, $|x|_{\mathcal{A}} := \inf_{z\in\mathcal{A}} |x - z|$.

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**Graph Theory Notation:** A directed graph (digraph) is defined as $\Gamma = (\mathcal{V}, \mathcal{E}, \mathcal{G})$. The set of nodes in the digraph is indexed by the elements of $\mathcal{V} := \{1, 2, \ldots, N\}$, and the edges are pairs in the set $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$. Each edge directionally links two nodes; i.e., an edge from $i$ to $k$ is denoted as $(i, k)$, which implies that agent $k$ is allowed to transmit information to agent $i$ across this link. The adjacency matrix of $\Gamma$ is defined by $\mathcal{G} = \{\gamma_{ik}\} \in \mathbb{R}^{N\times N}$, where $\gamma_{ik} = 1$ if $(i, k) \in \mathcal{E}$, and $\gamma_{ik} = 0$ otherwise. A digraph is undirected if $\gamma_{ik} = \gamma_{ki}$ for all $(i, k) \in \mathcal{E}$. The indegree and outdegree of agent $i$ are defined by $d_i^{in} = \sum_{k=1}^{N} \gamma_{ki}$ and $d_i^{out} = \sum_{k=1}^{N} \gamma_{ik}$, respectively. The adjacency set corresponding to the neighbors that can send information to the $i$-th agent is denoted by $\mathcal{J}_i := \{k \in \mathcal{V} : (k, i) \in \mathcal{E}\}$. A directed graph is strongly connected if it is possible to reach any node starting from any other node by traversing the directed edges. A digraph is weakly connected if there is an undirected path between any pair of vertices. A digraph is regular if each vertex has the same number of neighbors, i.e., every vertex has the same indegree and outdegree. For more details about graph theory and topologies see [15].

## II. Problem Description and Proposed Solution

### A. Problem Description

In this paper, we consider the problem of synchronization between the states of $N$ interconnected identical LTI systems. Specifically, the $i$-th node in the digraph is given by a system (or agent) with state $x_i$ having the following dynamics:

$$\dot{x}_i = Ax_i + Bu_i \qquad i \in \mathcal{V} := \{1, 2, \ldots, N\} \tag{1}$$

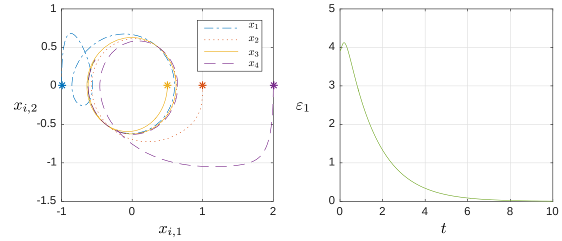
where $x_i \in \mathbb{R}^n$ is the state and $u_i \in \mathbb{R}^p$ is the input of the $i$-th system. The agents exchange information about their state variables intermittently over a directed graph $\Gamma = (\mathcal{V}, \mathcal{E}, \mathcal{G})$. Specifically, each agent receives information from its neighbors at isolated time instances given by the sequence of increasing times $\{t_s\}_{s=1}^{\infty}$, where $s \in \mathbb{N} \setminus \{0\}$ is the communication event index; namely, at each such $s$, the $i$-th agent receives the state of its $k$-th neighbor:

$$y_{ik}(t_s) = x_k(t_s) \tag{2}$$

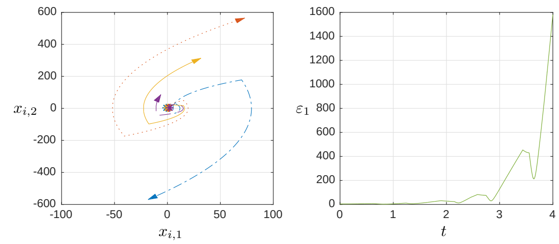
where $k \in \{k : (i, k) \in \mathcal{E}\}$. Moreover, given positive numbers $T_2 \geq T_1$, we assume that the continuous time elapsed between these events is governed by a discrete random variable with some bounded probability distribution. Namely, for each $s \in \mathbb{N} \setminus \{0\}$, the random variable $\Omega_s \in [T_1, T_2]$ determines the time elapsed between such communication events, i.e.,

$$t_{s+1} - t_s = \Omega_s \qquad \forall s \in \mathbb{N} \setminus \{0\}. \tag{3}$$

The scalar values $T_1$ and $T_2$ define the lower and upper bounds, respectively, of the time allowed to elapse between consecutive transmission instances. The value of the random variable $\Omega_s$ may be different for each $s \in \mathbb{N} \setminus \{0\}$ indicating nonperiodic update times. Furthermore, $\Omega_s$ may take values only on the bounded interval $[T_1, T_2]$, while the probability density function governing its distribution could be arbitrary.




(a) The error between the solutions using continuous control decreases to zero (right), implying that solutions synchronize asymptotically (left).




(b) With $t_{s+1} - t_s \in [0.3, 0.65]$, the intermittent communication prevents the controller from steering the distance between solutions to zero (right), and thus prevents synchronization (left).

Fig. 1. A numerical solution to (1) with controller assigning $u_i = K\sum_{k\in\mathcal{V}\setminus\{i\}}(x_i - x_k)$ with $K = [-1 \ \ -2]$ under (a) continuous communication and (b) intermittent communication. For each case, solutions from $((-1, 0), (1, 0), (0.5, 0), (2, 0))$ are given in planar plots as well as the error relative to $x_1$.

Our goal is to design a feedback controller assigning the inputs $u_i$ to drive the solutions to the family of systems as in (1) to synchronization, asymptotically; namely,

$$\lim_{t\to\infty} |x_i(t) - x_k(t)| = 0 \qquad \forall i, k \in \mathcal{V}$$

and also rendering stable the set of points $x = (x_1, x_2, \ldots, x_N)$ such that $x_i = x_k$ for each $i, k \in \mathcal{V}$.

Since the state of each system is available to its neighbors only at not necessarily periodic isolated time instances, it makes the design of the controller challenging. To illustrate this difficulty, consider four harmonic oscillators

$$\dot{x}_i = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x_i + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_i \tag{4}$$

for each $i \in \mathcal{V} = \{1, 2, 3, 4\}$ having state $x_i = (x_{i,1}, x_{i,2}) \in \mathbb{R}^2$, and input $u_i \in \mathbb{R}$. Suppose the digraph is completely connected.[1] Consider the case of a communication network providing continuous measurements of $x_i$'s and the static controller

$$u_i = K \sum_{k\in\mathcal{V}\setminus\{i\}} (x_i - x_k) \tag{5}$$

for each $i \in \mathcal{V}$. It can be shown that if there exists $K \in \mathbb{R}^{1\times 2}$ such that the matrix $A + 4BK$ is Hurwitz, then every solution to the closed-loop system asymptotically converges to synchronization. Figure 1(a) shows a numerical simulation for $K = \begin{bmatrix} -1 & -2 \end{bmatrix}$ and initial condition for $x$ equal to $((-1, 0), (1, 0), (0.5, 0), (2, 0))$. This figure also shows the norm of the error relative to $x_1$, i.e., $\varepsilon_1 =$

[1]A digraph is completely connected if every pair of distinct vertices is connected by a pair of unique edges (one in each direction).

$\left|\sum_{k\in\mathcal{V}\setminus\{1\}}(x_1-x_k)\right|$, along the solution which nicely converges to zero, i.e., synchronization is achieved. On the other hand, in Figure 1(b) we show a sample-and-hold implementation of the controller in (4) which is updated locally upon the reception of new state information. For the chosen arrival of information, occurring according to the update times $\{t_s\}_{s=1}^{\infty}$ with $t_{s+1}-t_s \in [0.3, 0.65]$ and with $K = \begin{bmatrix}-1 & -2\end{bmatrix}$ this implementation is not capable of synchronizing the systems. While an arbitrarily fast transmission rate would solve the problem, we show in Section III that our controller design synchronizes the systems with this "slow" transmission rate by properly designing $K$.

### B. Outline of Proposed Controller Design

In this paper, we propose a hybrid controller and a design procedure for synchronization of (1) over networks with intermittent transmission of information. The proposed controller has hybrid dynamics, and as such, it has state $\eta_i \in \mathbb{R}^p$, which assigns the input in (1) as $u_i = \eta_i$ for each $i$-th system. The controller is allowed to evolve continuously between the arrival of information, i.e., between the time instances $\{t_s\}_{s=1}^{\infty}$. Once new information from the neighboring systems is received, the controller uses this information to impulsively update its state to achieve synchronization. Due to the nonlinearities, nonperiodic arrival of information, and impulsive dynamics, classical control design tools provide little insight into solving this problem. This motivates us to design the controller by recasting the linear model (1), the impulsive network, and the hybrid controller in a hybrid system framework; precisely, the framework in [16].

### C. Hybrid Modeling

Due to the impulsive and nonperiodic nature of the communication structure across the network, we employ the network model proposed in [17]. More precisely, we capture these events via a timer variable $\tau$ which decreases with ordinary time and, upon reaching zero, is impulsively reset to a point within the interval $[T_1, T_2]$. To model this mechanism and the closed-loop system, we employ the hybrid systems framework in [16], where a hybrid system is given by four objects $(C, F, D, G)$ defining its *data*:

- *Flow set*: a set $C \subset \mathbb{R}^n$ specifying the points where the continuous evolution (or flows) is possible;
- *Flow map*: a set-valued map $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ defining the flows;
- *Jump set*: a set $D \subset \mathbb{R}^n$ specifying the points where the discrete evolution (or jumps) is possible;
- *Jump map*: a set-valued map $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ defining the value of the state after jumps.

Then, the hybrid system with state $\zeta \in \mathbb{R}^n$ is denoted by $\mathcal{H} = (C, F, D, G)$ and can be written in compact form as

$$\mathcal{H}: \qquad \zeta \in \mathbb{R}^n \quad \begin{cases} \dot{\zeta} \in F(\zeta) & \zeta \in C \\ \zeta^+ \in G(\zeta) & \zeta \in D \end{cases}. \tag{6}$$

Using this framework, the evolution of the timer $\tau$ modeling the events is given by

$$\begin{aligned} \dot{\tau} &= -1 & \tau &\in [0, T_2], \\ \tau^+ &\in [T_1, T_2] & \tau &= 0. \end{aligned} \tag{7}$$

The reset law of the timer is set valued and it resets $\tau$ to any value within the interval $[T_1, T_2]$. Due to this fact, any sequence of points $\{t_s\}_{s=1}^{\infty}$ that satisfies (3) with $\Omega_s$ given by any bounded probability function on the interval $[T_1, T_2]$ is captured by (7).

The network dynamics only allow for information exchange at isolated events. This leads us to design a controller that evolves continuously between such events and updates its state impulsively when new information arrives. More precisely, we consider a generic decentralized state-feedback hybrid controller with state $\eta_i$ and dynamics[2]

$$\begin{aligned} \dot{\eta}_i &= F_{ci}(x_i, \eta_i) & \tau &\in [0, T_2] \\ \eta_i^+ &= \sum_{k\in\mathcal{J}_i} G_{ci}^k(x_i, \eta_i, x_k) & \tau &= 0 \end{aligned} \tag{8}$$

where $F_{ci} : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^p$ defines the continuous evolution of the controller and $G_{ci}^k : \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^n \to \mathbb{R}^p$ defines the impulsive update law, which uses state-based information from its neighbors defined by the set $\mathcal{J}_i$.

The closed-loop hybrid system, denoted $\mathcal{H}$, with its state defined as $\zeta = (\xi, \tau) = \mathbb{R}^{N(n+p)} \times [0, T_2] =: \mathcal{X}$, where $\xi = (\xi_1, \xi_2, \ldots, \xi_N)$ and, for each $i \in \mathcal{V}$, $\xi_i$ is partitioned as $(x_i, \eta_i) \in \mathbb{R}^n \times \mathbb{R}^p$. The input $u_i$ to (1) is assigned to $\eta_i$. The flow and jump sets are defined to constrain the evolution of the timer. From the timer model in (7), the controller dynamics in (8), and the LTI plant in (1), the flow set of $\mathcal{H}$ is defined as the set that constrains the timer states to the interval $[0, T_2]$. Then, we have that the flow set and flow map are given by

$$C := \mathcal{X} \tag{9}$$

$$F(\zeta) := (F_1(\xi_1), F_2(\xi_2), \ldots, F_N(\xi_N), -1) \quad \forall \zeta \in \mathcal{X} \tag{10}$$

respectively, where

$$F_i(\xi_i) = \begin{bmatrix} Ax_i + B\eta_i \\ F_{ci}(x_i, \eta_i) \end{bmatrix}$$

for each $i \in \mathcal{V}$. The impulsive events are captured by the jump set $D$ and the jump map $G$. Since jumps are only triggered upon the arrival of information, i.e., when $\tau = 0$, the jump set is defined as

$$D = \{\zeta \in \mathcal{X} : \tau = 0\}. \tag{11}$$

Then, the jump map is defined as

$$G(\zeta) = (G_1(\xi), G_2(\xi), \ldots, G_N(\xi), [T_1, T_2]) \quad \forall \zeta \in D \tag{12}$$

where, for each $i \in \mathcal{V}$, $G_i$ is defined as

$$G_i(\xi) = \begin{bmatrix} x_i \\ \sum_{k\in\mathcal{J}_i} G_{ci}^k(x_i . \eta_i, x_k) \end{bmatrix}.$$

*Lemma 2.1: If, for each $i \in \mathcal{V}$, $F_{ci} : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^p$ and $G_{ci}^k : \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^n \to \mathbb{R}^p$ are continuous, then the hybrid system $\mathcal{H}$ with data given by* (9)-(12) *satisfies the hybrid*

[2]This hybrid systems framework also allows for set-valued controller dynamics.

*basic conditions.*[3]

Note that satisfying the hybrid basic conditions implies that $\mathcal{H}$ is well-posed and, with asymptotic stability of a compact set (defined in Section III in the appropriate coordinates), automatically gives robustness to small enough perturbations; see [16] for more information.

A solution $\phi$ to $\mathcal{H}$ is parametrized by $(t, j) \in \mathbb{R}_{\geq 0} \times \mathbb{N}$, where $t$ denotes ordinary time and $j$ denotes jump time. The domain $\operatorname{dom} \phi \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ is a hybrid time domain if, for every $(T, J) \in \operatorname{dom} \phi$, the set $\operatorname{dom} \phi \cap ([0, T] \times \{0, 1, \dots, J\})$ can be written as $\cup_{j=0}^{J} (I_j \times \{j\})$, where $I_j = [t_j, t_{j+1}]$ for a time sequence $0 = t_0 \leq t_1 \leq \cdots \leq t_J \leq t_{J+1}$. The $t_j$'s with $j > 0$ define the time instants when the state of the hybrid system jumps and $j$ counts the number of jumps.

*Remark 2.2:* Due to the fact that the timer variable reaching zero is what triggers the jumps, some properties of the domain of solutions can easily be characterized. In particular, a solution $\phi$ to the hybrid system $\mathcal{H}$ is such that, with $t_0 = 0$, $T_1 \leq t_{j+1} - t_j \leq T_2$ for all $j \geq 2$, and $0 \leq t_1 \leq T_2$, which lead to

$$(j-1)T_1 \leq t \leq (j+1)T_2 \tag{13}$$

for all $(t, j) \in \operatorname{dom} \phi$.

Our approach to show synchronization is to define a set stabilization problem. In fact, we determine the attractivity and stability of a set, denoted $\mathcal{A}$, enforcing that the appropriate state components of the solutions to the resulting closed-loop hybrid system are equal. For this purpose, we employ the following notion of global exponential stability of a set for general hybrid systems.

*Definition 2.3:* Consider a hybrid system $\mathcal{H}$ on $\mathbb{R}^n$. Let $\mathcal{A} \subset \mathbb{R}^n$ be closed. The set $\mathcal{A}$ is *globally exponentially stable (GES)* for $\mathcal{H}$ if there exists $\kappa, r > 0$ such that every maximal solution $\phi$ to $\mathcal{H}$ is complete and satisfies, for all $(t, j) \in \operatorname{dom} \phi$, $|\phi(t, j)|_{\mathcal{A}} \leq r e^{-\kappa(t+j)} |\phi(0, 0)|_{\mathcal{A}}$.

With an appropriate definition of the set $\mathcal{A}$, the notion in Definition 2.3 corresponds to a synchronization property for $\mathcal{H}$. At times, when such a property holds, we say that the hybrid system $\mathcal{H}$ exponentially synchronizes.

## III. MAIN RESULT

### A. Strongly Connected Network

We consider a sample-and-hold state-feedback instance of the controller in (8), namely, we pick $F_{ci}(x_i, \eta_i) = 0$ for each $i \in \mathcal{V}$ and $G_{ci}^{k}(x_i, \eta_i, x_k) = \frac{K_i}{d_i^{in}}(x_i - x_k)$ for each $(i, k) \in \mathcal{E}$. Building from (8), the controller is as follows:

$$\begin{aligned} \dot{\eta}_i &= 0 && \tau \in [0, T_2] \\ \eta_i^{+} &= \frac{K_i}{d_i^{in}} \sum_{k \in \mathcal{J}_i} (x_i - x_k) && \tau = 0 \end{aligned} \tag{14}$$

[3]A hybrid system $\mathcal{H} = (C, F, D, G)$ is said to satisfy the hybrid basic conditions if the sets $C$ and $D$ are closed, the function $F : \mathbb{R}^n \to \mathbb{R}^n$ is continuous and the set-valued mapping $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is outer semicontinuous and locally bounded relative to $D$, and $D \subset \operatorname{dom} G$. A set-valued mapping $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is *outer semicontinuous* if its graph $\{(x, y) : x \in \mathbb{R}^n, y \in G(x)\}$ is closed; see [16, Lemma 5.10].

where $K_i \in \mathbb{R}^{n \times p}$ is the local gain matrix and $d_i^{in}$ is the indegree[4] for the $i$-th system. We define the set to stabilize as $\mathcal{A} := \{\zeta \in \mathcal{X} : x_i = x_k, \eta_i = 0 \ \forall i, k \in \mathcal{V}, \tau \in [0, T_2]\}$ which is closed. Note that, due to the definition of the controller, when the systems are synchronized, i.e., when $x_i = x_k$ for each $i, k \in \mathcal{V}$, then for each $i \in \mathcal{V}$ the controller state for each agent is updated to zero and from the definition of the flow map for $\eta_i$ it stays at zero between jumps.

To formulate conditions guaranteeing exponential stability of $\mathcal{A}$, we change the coordinates of $\mathcal{H}$. Denote the resulting system as $\mathcal{H}_\varepsilon^{sc} = (C_\varepsilon^{sc}, F_\varepsilon^{sc}, D_\varepsilon^{sc}, G_\varepsilon^{sc})$, where the superscript 'sc' stands for 'strongly connected.' This new system has state $\chi = (z, \tau) \in \mathcal{X}$ with $z = (z_1, z_2, \dots, z_N)$ and $z_i = (\varepsilon_i, \eta_i)$, where $\varepsilon_i$ is the local relative error of the $i$-th system given by

$$\varepsilon_i = \frac{1}{d_i^{in}} \sum_{k \in \mathcal{J}_i} (x_i - x_k). \tag{15}$$

For each $i \in \mathcal{V}$, we have that when $\tau \in [0, T_2]$, the change of $\varepsilon_i$ and $\eta_i$ along flows is given by $\dot{\varepsilon}_i = A\varepsilon_i + B\eta_i - \frac{B}{d_i^{in}} \sum_{k \in \mathcal{J}_i} \eta_k$, while, as in (14), $\dot{\eta}_i = 0$. Likewise, when $\tau = 0$, communication instantaneously occurs between all agents, and the discrete dynamics in the error coordinates are given by $\varepsilon_i^{+} = \varepsilon_i$ and $\eta_i^{+} = K_i \varepsilon_i$. Therefore, the new coordinates leads to a hybrid system $\mathcal{H}_\varepsilon^{sc}$ with the following data:

$$\begin{aligned} F_\varepsilon^{sc}(\chi) &:= \begin{bmatrix} \bar{A}_f z \\ -1 \end{bmatrix} && \forall \chi \in C_\varepsilon^{sc} := \mathcal{X} \\ G_\varepsilon^{sc}(\chi) &:= \begin{bmatrix} \bar{A}_g z \\ [T_1, T_2] \end{bmatrix} && \forall \chi \in D_\varepsilon^{sc} := \{\chi \in \mathcal{X} : \tau = 0\} \end{aligned} \tag{16}$$

where

$$\begin{aligned} \bar{A}_f &= I \otimes A_f - (\mathcal{G}^\top D_{in}^{-1}) \otimes B_f \\ \bar{A}_g &= \operatorname{diag}(A_{g1}, A_{g2}, \dots, A_{gN}) \end{aligned}$$

with $D_{in} = \operatorname{diag}(d_1^{in}, d_2^{in}, \dots, d_N^{in})$, and

$$A_f = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}, \ B_f = \begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix}, \ A_{gi} = \begin{bmatrix} I & 0 \\ K_i & 0 \end{bmatrix}. \tag{17}$$

For the hybrid system $\mathcal{H}_\varepsilon^{sc}$, we want to stabilize the set given by

$$\mathcal{A}_\varepsilon := \{\chi \in \mathcal{X} : z = 0, \tau \in [0, T_2]\} \tag{18}$$

which is compact. Next, we present sufficient conditions for exponential stability of the set $\mathcal{A}_\varepsilon$ for $\mathcal{H}_\varepsilon^{sc}$.

*Theorem 3.1: Let $T_1$ and $T_2$ be positive scalars such that $T_1 \leq T_2$, and the digraph $\Gamma$ be strongly connected. Suppose that there exist a positive scalar $\sigma$, a positive definite symmetric matrix $P \in \mathbb{R}^{N(n+p) \times N(n+p)}$, and for each $i \in \mathcal{V}$, a matrix $K_i \in \mathbb{R}^{p \times n}$ such that*

$$e^{\sigma \nu} \bar{A}_g^\top e^{\bar{A}_f^\top \nu} P e^{\bar{A}_f \nu} \bar{A}_g - P < 0 \qquad \forall \nu \in [T_1, T_2]. \tag{19}$$

*Then, the set $\mathcal{A}_\varepsilon$ in (18) is GES for $\mathcal{H}_\varepsilon^{sc}$ in (16).*

*Remark 3.2:* Note that condition (19) is akin to the

[4]Recall that $d_i^{in}$ is the indegree of agent $i$ defined as $\sum_{k=1}^{N} \gamma_{ki}$ where $\gamma_{ki}$ are the elements of the adjacency matrix $\mathcal{G}$.

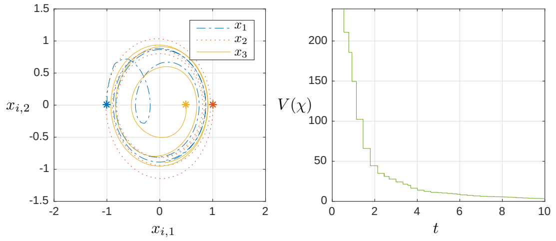



Fig. 2. The $(x_1, x_2, x_3)$ components of a numerical solution for Example 3.4 (left). The Lyapunov function $V$ along a solution to $\mathcal{H}^{sc}_\varepsilon$ decreases to zero, indicating that the solution synchronizes over hybrid time. (right)

discrete Lyapunov equation with system matrix $H(\nu) = e^{\sigma\nu/2} e^{\bar{A}_f \nu} \bar{A}_g$. Furthermore, this fact implies that condition (19) is satisfied if, for all $\nu \in [T_1, T_2]$, the eigenvalues of $H(\nu)$ are contained within the unit circle.

*Remark 3.3:* The form of condition (19) may be difficult to satisfy numerically. In fact, this condition is not convex in $K_i$ and $P$, and needs to be verified for infinitely many values of $\nu$. In [17], the authors use a polytopic embedding strategy, in which one needs to find matrices $X_i$ such that the exponential matrix is an element in the convex hull of the $X_i$ matrices so as to solve a linear matrix inequality. These results can be adapted to our setting.

*Example 3.4:* Consider the case of a network of three harmonic oscillators in (4) connected on a graph with adjacency matrix

$$\mathcal{G} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

with $T_1 = 0.13$ and $T_2 = 0.35$. By applying Theorem 3.1 to this network, it follows that parameters $K_1 = \begin{bmatrix} -0.4 & -1 \end{bmatrix}$, $K_2 = \begin{bmatrix} -0.5 & -0.2 \end{bmatrix}$, $K_3 = \begin{bmatrix} -0.2 & -0.15 \end{bmatrix}$, $\sigma = 0.1$ and $P$ given by

$$P \approx \left[ \begin{matrix} 65.6 & 16.0 & 3.0 & -8.6 & 8.1 \\ 16.0 & 52.3 & 5.0 & -13.2 & -2.6 \\ 3.0 & 5.0 & 11.0 & -1.4 & -0.4 \\ -8.6 & -13.2 & -1.4 & 67.3 & 2.4 \\ 8.1 & -2.5 & -0.4 & 2.4 & 41.4 \\ -1.1 & -1.8 & -4.8 & 6.5 & 1.0 \\ -10.0 & -3.8 & -0.6 & -26.5 & 2.5 \\ 2.1 & -5.6 & -0.2 & -7.6 & -17.3 \\ -1.8 & -3.1 & -6.1 & -5.0 & -0.5 \end{matrix} \right. \cdots$$

$$\cdots \left. \begin{matrix} -1.1 & -10.0 & 2.1 & -1.8 \\ -1.8 & -3.8 & -5.6 & -3.1 \\ -4.8 & -0.6 & -0.2 & -6.1 \\ 6.5 & -26.5 & -7.6 & -5.0 \\ 1.0 & 2.6 & -17.3 & -0.5 \\ 14.9 & -2.3 & -1.0 & -10.1 \\ -2.3 & 14.7 & 1.6 & 2.9 \\ -1.0 & 1.6 & 10.1 & 1.3 \\ -10.1 & 2.9 & 1.2 & 16.3 \end{matrix} \right] \quad (20)$$

satisfy condition (19). Figure 2 shows a numerical solution $\phi = (\phi_x, \phi_\eta, \phi_\tau)$ for the hybrid system $\mathcal{H}$ from $\phi_x(0,0) = (-1, 0, 1, 0, 0.5, 0)$, $\phi_\eta(0,0) = (1, 2, -1)$ and $\phi_\tau(0,0) = 0.1$. In this figure, we have a plot of solutions on the $(x_{i,1}, x_{i,2})$–plane of each $x_i$ projected on to flow time and the Lyapunov function $V(\chi) = e^{\sigma\tau} z^\top e^{\bar{A}_f^\top \tau} P e^{\bar{A}_f \tau} z$ evaluated along the solution (in error coordinates).[5]

[5]Code at `https://github.com/HybridSystemsLab/LTISyncStrong`

### *B. Completely Connected Network*

In this section, we consider the case of a completely connected graph; namely, for any two nodes in the graph, there exists an edge in both directions in which communication occurs synchronously over the entire network. Replacing $d_i^{in}$ by $N$ and $\mathcal{J}_i$ by $\mathcal{V} \setminus \{i\}$ in the controller used in (14), we arrive to the following controller:

$$\begin{aligned} \dot{\eta}_i &= 0 & \tau &\in [0, T_2] \\ \eta_i^+ &= \frac{K}{N} \sum_{k \in \mathcal{V}\setminus\{i\}} (x_i - x_k) & \tau &= 0 \end{aligned} \quad (21)$$

with $K \in \mathbb{R}^{p \times n}$ being the controller gain matrix common for each agent.

To design $K$ for exponential stability of $\mathcal{A}$, we change coordinates of $\mathcal{H}$. This system in the new coordinates is denoted $\mathcal{H}^c_\varepsilon$ where 'c' stands for 'completely connected.' The error coordinates are $\chi = (z, \tau) \in \mathcal{X}$, where $z = (z_1, z_2, \ldots, z_N)$ and $z_i = (\varepsilon_i, \mu_i)$ with

$$\begin{aligned} \varepsilon_i &:= \frac{1}{N-1} \sum_{k \in \mathcal{V}\setminus\{i\}} (x_i - x_k) \\ \mu_i &:= \frac{1}{N-1} \sum_{k \in \mathcal{V}\setminus\{i\}} (\eta_i - \eta_k), \end{aligned} \quad (22)$$

respectively. Due to the symmetry of the digraph, we have that when $\tau \in [0, T_2]$, the continuous evolution of the error dynamics are given by $\dot{\varepsilon}_i = A\varepsilon_i + B\mu_i$, $\dot{\mu}_i = 0$ for each $i \in \mathcal{V}$. Likewise, when $\tau = 0$, the discrete dynamics reduce to $\varepsilon_i^+ = \varepsilon_i$ and $\mu_i^+ = K\varepsilon_i$. Then, the hybrid closed-loop system $\mathcal{H}^c_\varepsilon$ has data given by

$$\begin{aligned} F^c_\varepsilon(\chi) &= \begin{bmatrix} (I \otimes A_f) z \\ -1 \end{bmatrix} & \forall \chi &\in C^c_\varepsilon := \mathcal{X}, \\ G^c_\varepsilon(\chi) &= \begin{bmatrix} (I \otimes A_g) z \\ [T_1, T_2] \end{bmatrix} & \forall \chi &\in D^c_\varepsilon := \{\chi \in \mathcal{X} : \tau = 0\} \end{aligned} \quad (23)$$

where $A_f$ is given in (17) and $A_g = \begin{bmatrix} I & 0 \\ K & 0 \end{bmatrix}$.

The following result provides conditions guaranteeing that $\mathcal{A}_\varepsilon$ in (18) is GES for the hybrid system $\mathcal{H}^c_\varepsilon$, namely, that $\mathcal{H}$ exponentially synchronizes.

*Theorem 3.5: Let $T_1$ and $T_2$ be positive scalars such that $T_1 \le T_2$, and the digraph $\Gamma$ be completely connected. Suppose that there exist a positive scalar $\sigma$, a positive definite symmetric matrix $P \in \mathbb{R}^{(n+p)\times(n+p)}$, and a matrix $K \in \mathbb{R}^{p \times n}$ such that*

$$e^{\sigma\nu} A_g^\top e^{A_f^\top \nu} P e^{A_f \nu} A_g - P < 0 \qquad \forall \nu \in [T_1, T_2]. \quad (24)$$

*Then, the set $\mathcal{A}_\varepsilon$ in (18) is GES for $\mathcal{H}^c_\varepsilon$ in (23).*

*Example 3.6:* We apply Theorem 3.5 to the network of four harmonic oscillators in (4), with $T_1 = 0.3$ and $T_2 = 0.65$ and obtain parameters $P \approx \begin{bmatrix} 23.6 & 5.11 & 3.30 \\ 5.11 & 13.61 & 1.84 \\ 3.30 & 1.84 & 7.51 \end{bmatrix}$, $K = \begin{bmatrix} -0.5 & -0.7 \end{bmatrix}$, and $\sigma = 0.1$. Figure 3 shows a numerical solution $\phi = (\phi_x, \phi_\eta, \phi_\tau)$ for the hybrid system

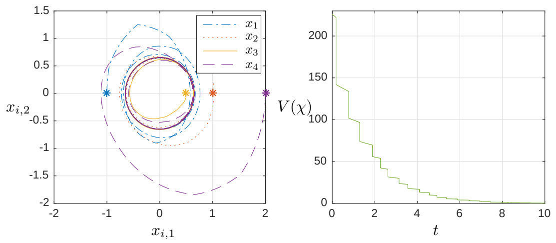



Fig. 3. The $(x_1, x_2, x_3, x_4)$ components of a solution to $\mathcal{H}$ in Example 3.6, which uses a completely connected graph. Due to the fact that the Lyapunov function $V$ along a solution to $\mathcal{H}_\varepsilon^c$ is decreasing to zero (right), indicates that the solution synchronizes over time (left).

$\mathcal{H}$ with a completely connected digraph from $\phi_x(0,0) = ((-1,0),(1,0),(0.5,0),(2,0))$, $\phi_\eta(0,0) = (1.7,2,0,1.7)$ and $\phi_\tau(0,0) = 0.2$. In this figure, we have a $(x_{i,1}, x_{i,2})$–planar plot of the $x_i$ states and the Lyapunov function $V(\chi) = \sum_{i\in\mathcal{V}} e^{\sigma\tau} z_i^\top e^{A_f^\top \tau} P e^{A_f \tau} z_i$ evaluated along the solution (in error coordinates).[6] Recalling the motivational example in Section II-A, this controller design solves the problem of synchronization with "slow" transmission times.

### C. Robustness to Communication Noise

In a realistic setting, the information transmitted would be subjected to some amount of noise. In this section, we consider the hybrid system $\mathcal{H}_\varepsilon^c$ in Section III-B under the effect of communication noise $m_k \in \mathbb{R}^n$ induced by perturbations in the communication network connecting the systems. Specifically, the $i$-th system receives the state of the $k$-th system corrupted by $m_k \in \mathbb{R}^n$, namely, for each $i \in \mathcal{V}$, we have that $y_{ik} = x_k + m_k$. In such a case, the update law of the controller state in (21) becomes

$$\begin{aligned} \dot{\eta}_i &= 0 & \tau &\in [0, T_2] \\ \eta_i^+ &= \frac{K}{N} \sum_{k\in\mathcal{V}\setminus\{i\}} (x_i - y_{ik}) & \tau &= 0 \end{aligned} \tag{25}$$

which, different from (21), receives the state plus noise. The update law becomes $\eta_i^+ = \frac{K}{N}\sum_{k\in\mathcal{V}\setminus\{i\}}(x_i - x_k) - K\bar{m}_i$ where $\bar{m}_i = \frac{1}{N}\sum_{k\in\mathcal{V}\setminus\{i\}} m_k$. Furthermore, through the change in coordinates $\mu_i$ for $\mathcal{H}_\varepsilon^c$ as in (22), we have that at jumps $\mu_i^+ = K\varepsilon_i - K\tilde{m}_i$, where $\tilde{m}_i = \frac{1}{N-1}\sum_{k\in\mathcal{V}\setminus\{i\}}(\bar{m}_i - \bar{m}_k)$. As amplification of the noise may occur at jumps, inspired by [17], we show that the hybrid system in Section III-B is input-to-state (ISS) stable[7] with respect to the measurement noise $\tilde{m}_i$.

*Theorem 3.7: Let $T_1$ and $T_2$ be two positive scalars such that $T_1 \le T_2$ and the digraph $\Gamma$ be strongly connected. If there exist a positive scalar $\sigma$, a symmetric positive definite matrix $P \in \mathbb{R}^{(n+p)\times(n+p)}$ and a matrix $K \in \mathbb{R}^{p\times n}$ satisfying* (24)*, then the hybrid system $\mathcal{H}_\varepsilon^c$ with data in* (23) *with $\mu_i^+ = K\varepsilon_i - K\tilde{m}_i$ is ISS with respect to $\tilde{m} = (\tilde{m}_1, \ldots, \tilde{m}_N)$ relative to the set $\mathcal{A}_\varepsilon$ in* (18).

[6] Code at `https://github.com/HybridSystemsLab/LTISyncComplet`

[7] Input-to-state stability is defined as follows: given a compact set $\mathcal{A} \subset \mathbb{R}^n$, the hybrid system $\mathcal{H}$ is ISS with respect to $u$ relative to $\mathcal{A}$ if there exists $\beta \in \mathcal{KL}$ and $\kappa \in \mathcal{K}$ such that, for each $\xi \in \mathbb{R}^n$, each maximal $(\phi, u)$ with $\phi(0,0) = \xi$, satisfies $|\phi(t,j,\xi,u)|_\mathcal{A} \le \max\{\beta(|\xi|_\mathcal{A}, t+j), \kappa(||u||_{(t,j)})\}$ for each $(t,j) \in \mathrm{dom}(\phi, u)$.

*Remark 3.8:* Since the hybrid systems $\mathcal{H}_\varepsilon^{sc}$ and $\mathcal{H}_\varepsilon^c$ satisfy the hybrid basic conditions and have a compact set $\mathcal{A}_\varepsilon$ asymptotically stable, the stability of $\mathcal{A}_\varepsilon$ is robust to general small perturbations. See [16] for more information.

## IV. Conclusion

In this paper, we showed that hybrid sample-and-hold state-feedback controllers are viable algorithms for synchronizing $N$ linear time-invariant systems with stochastic transmission events. The communication network between the systems was modeled by a decreasing timer that is reset to some point in a bounded interval, which allowed us to allow for arbitrary probability distributions triggering the transmission events. Recasting synchronization as a set stabilization problem, Lyapunov functions were constructed to certify asymptotic stability of this set, implying that the networked systems synchronize asymptotically. The results in this paper can be used to design large-scale networked systems that communicate at stochastic instants over other generalized graphs.

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