Synchronization of Two Linear Systems over Intermittent Communication Networks with Robustness

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Abstract— **The property of synchronization in a network of two identical linear time invariant (LTI) systems connected through a network with sporadic communication is studied. Under this network, the goal of the systems is to synchronize the values of their state while communication occurs only at stochastically-driven isolated time instances. To solve this problem, we propose a dynamic controller with both continuous dynamics between communication times and impulsive dynamics when information is received. Due to the continuous and discrete dynamics inherent to the networked systems and communication structure, we use a hybrid systems framework to model the closed-loop system. The problem of synchronization is then recast as a set stabilization problem and, by utilizing recent Lyapunov stability tools for hybrid systems, sufficient conditions for asymptotic stability of the synchronization set are provided for two network topologies: a cascade (unidirectional) network, and a feedback (bidirectional) communication network with independent transmission instances. Furthermore, robustness of synchronization is studied for two classes of perturbations on the states of the system as well as skews in the clocks triggering the events. Numerical examples are given to illustrate the main results.**

I. Introduction

Networked systems have many real-world applications such as large-scale sensor networks, genetic regulatory networks, neuronal networks, smart grids, etc, [1], [2], [3]. The goals of many engineering and natural systems can be formulated in terms of synchronization-type properties, which has become an increasingly popular [4], [5], [6], [7]. While similar to consensus and coordination, synchronization emphasizes on the dynamics of the individual systems rather than the network topology and its constraints. Specifically, the system dynamics are allowed to be modified (via control strategies or coupling terms) through information exchange to guarantee that the distance between the state of each system approaches zero [8], [9], [10]. Furthermore, in the absence of a controlled communication structure, the individual systems can be unstable or even chaotic.

The idea of synchronization in networked systems has been extensively studied in the literature and has been approached by various viewpoints and methodologies. Synchronization of coupled identical linear systems has been widely investigated in both the continuous and discrete time domains [4], [5], [11]. Further studies involving nonlinear systems within complex network structures are also available, [12]. A typical approach is to study the network structure using graph theory, which provides a solid understanding of the connectivity of the network and its effect on the individual dynamics of the systems [13], [14]. The study of the stability of synchronization using systems theory tools, like Lyapunov functions [15], [16], contraction theory [17], and incremental input-to-state stability [18], [19] have also been explored. Synchronization of state information in impulsive networks with sporadic communication between LTI systems naturally leads to a complex communication structure, for which, perhaps due to a theory for synchronization of hybrid systems not being available, there is a notable lack of solutions. Notably, recent research efforts on sample-data systems [20] and triggered control [21] for the stabilization of sets provide results that can become useful for synchronization, though some of the assumptions need to be carefully fit to the synchronization problem under intermittent communication networks; e.g., the common event times condition required in [22] might not be always satisfied in such networks.

This article deals with the problem of synchronization of the states between two continuous-time systems when communication occurs only impulsively, at stochastically determined instants. To solve this problem, we design a dynamic hybrid controller that continuously evolves between communication times and undergoes an instantaneous change in its state when a new measurement is available. Due to the combination of continuous and impulsive dynamics, we use hybrid systems theory to model the systems, the controllers, and two network topologies, and apply tools for the stabilization of sets. More precisely, we define the notion of uniform global asymptotic synchronization for hybrid systems and specialize it to the problem of study under a cascade topology and a feedback topology. We recast the synchronization problem as the stabilization of a compact set and apply a Lyapunov theorem for hybrid systems. We show that the resulting closed loop system has an asymptotic synchronization property that is robust to state perturbations, general unmodeled dynamics, and, in particular, to skewed clocks triggering the communication events.

The remainder of this paper is organized as follows. In Section II we introduce the impulsive network model, control structure and recast this model into the hybrid systems framework. In Section III, we provide the sufficient conditions that guarantee stability for each topology. The results on robustness of synchronization are in Section III-C. Section IV provides examples and numerical simulations.

Notation: The set $\mathbb{R}$ denotes the space of real numbers. The set $\mathbb{R}^n$ denotes the n-dimensional Euclidean space.

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The set $\mathbb{N}$ denotes the natural numbers including zero, i.e., $\mathbb{N} = \{0, 1, 2, ...\}$. Given two vectors x, y, we denote $(x, y) = [x^\top, y^\top]^\top$. Given $x \in \mathbb{R}^n$, $|x|$ denotes the Euclidean norm of x. The set $\mathbb{B}$ is the closed unit ball centered around the origin in Euclidean space. The identity matrix is denoted by I. For a matrix $A \in \mathbb{R}^{n\times m}$, $A^\top$ denotes the transpose of A and $|A|$ denotes the induced 2-norm. For two symmetric matrices with same dimensions, A and B, $A > B$ means that $A - B$ is positive definite. A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to belong to class $\mathcal{K}$ if is continuous, zero at zero, and strictly increasing and is a class $\mathcal{K}_\infty$ function if it belongs to class $\mathcal{K}$ and is unbounded. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to belong to class $\mathcal{KL}$ if it is nondecreasing in its first argument, non-increasing in its second argument, and is such that $\lim_{s\to 0} \beta(s, t) = \lim_{t\to\infty} \beta(s, t) = 0$. Given a closed set $\mathcal{A} \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$, $|x|_\mathcal{A} := \inf_{z\in\mathcal{A}} |x - z|$.

II. Problem Description and Mathematical Modeling

A. Problem Description

In this paper, we consider the problem of synchronizing the states x_1 and x_2 of a pair of continuous-time systems given by

$$\begin{aligned} \dot{x}_1 &= Ax_1 + Bu_1, \qquad y_1 = x_1 \\ \dot{x}_2 &= Ax_2 + Bu_2, \qquad y_2 = x_2 \end{aligned} \tag{1}$$

exchanging information about their variables intermittently, where, for each $i \in \{1, 2\}$, $x_i \in \mathbb{R}^n$ is the state, $y_i \in \mathbb{R}^q$ is the output, and $u_i \in \mathbb{R}^p$ is the input to the i-th system. More precisely, our goal is to design a feedback controller assigning the inputs u_1 and u_2 to drive the solutions to (1) to synchronization between x_1 and x_2 asymptotically and also rendering the set of points where x_1 is equal to x_2 stable. Moreover, the outputs of each system are available to each other only at isolated time instances. To accommodate many real-world applications, we do not assume that information arrival to each system occurs simultaneously. Specifically, we assume that the output y_j of the j-th system is available to the i-th system ($i, j \in \{1, 2\}$, $i \neq j$) only at impulsive time instances t^i_k, where $k \in \mathbb{N} \setminus \{0\}$ is the communication event index and $i \in \{1, 2\}$ is the index denoting the system receiving information at such instants. Given positive numbers $T_2 > T_1$, we assume that the time between these events are governed by a discrete random variable with some bounded probability distribution: for each $i \in \{1, 2\}$ the random variable $\chi^i \in [T_1, T_2]$ determines the time elapsed between such events, namely,

$$t^i_{k+1} - t^i_k = \chi^i \qquad \forall k \in \mathbb{N} \setminus \{0\} \tag{2}$$

The scalar values T_1 and T_2 define the lower and upper bounds of the time allowed to elapse between consecutive transmission instances. In this way, the random variable χ^i may take values only on the bounded interval $[T_1, T_2]$, while the probability density function governing its distribution can be arbitrary.

Synchronization itself is most generally described as the property that the distance between every pair of solutions, one to each system, converges to zero. At times, stability of the synchronization condition is also required. More precisely, for the continuous-time system in (1), let $\phi = (\phi_1, \phi_2)$ be a solution to (1) from $\phi(0) = (\phi_1(0), \phi_2(0))$, namely, ϕ_1 is a solution to the system with state x_1 and, likewise, ϕ_2 for the system with state x_2. In this paper, due to the interest in robustness, we consider stability and attractivity of synchronization. Informally, the system is said to have

- *stable synchronization* if, for every $\epsilon > 0$, there exists $\delta > 0$ such that $|\phi_1(0) - \phi_2(0)| \leq \delta$ implies $|\phi_1(t) - \phi_2(t)| \leq \epsilon$ for all $t \geq 0$ and for every solution ϕ to (1) from $(\phi_1(0), \phi_2(0))$.
- *attractive synchronization* if every solution is defined on $[0, \infty)$ and satisfies
$$\lim_{t\to\infty} |\phi_1(t) - \phi_2(t)| = 0.$$
- *asymptotic synchronization* if it has both stable synchronization and attractive synchronization.

To illustrate that synchronization with intermittent information is difficult, consider a pair of linear oscillators, each with the dynamics

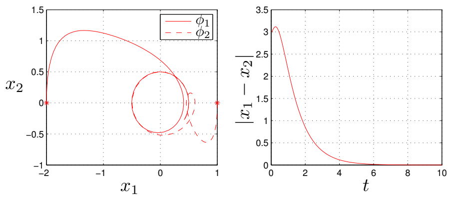
$$\dot{x}_i = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x_i + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_i, \tag{3}$$

with state $x_i \in \mathbb{R}^2$ and input $u_i \in \mathbb{R}$, for each $i \in \{1, 2\}$. It can be proven that for a network transmitting $y_i = x_i$ continuously, the static controller given by $u_i = K_i(y_i - y_j) = K_i(x_i - x_j)$ asymptotically synchronizes the system if there exist $K_1, K_2 \in \mathbb{R}^{1\times 2}$ such that the matrix $A+B(K_1+K_2)$ is Hurwitz. See Figure 1(a) for a numerical solution for $K_1 = K_2 = [-1, -2]$ and initial conditions $x_1 = (-2, 0)$ and $x_2 = (1, 0)$. The norm of the error between x_1 and x_2 along the solution is also plotted. As Figure 1(b) shows, a sample-and-hold implementation of this controller may not work over a network with intermittent information, in which information cannot be transmitted arbitrarily fast. Since information arrives only at impulsive events (in this simulation, we assume they occur at the same time instances for both system), the controller updates the input to the plant at such time instants which, for the arrival of information occurring with $t^1_{k+1} - t^1_k = t^2_{k+1} - t^2_k \in [.5, 1]$ for all k, is not capable of synchronizing the systems. While very fast transmission would solve the problem, we show in Section IV-A that our controller synchronizes the systems with this "slow" transmission rate.

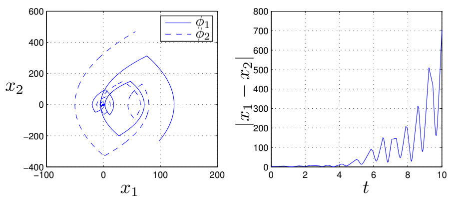
In this paper, we propose a hybrid controller and a design procedure for synchronization for intermittent networks. The proposed controller for each system has state η_i, which assigns the input as $u_i = \eta_i$, where $\eta_i \in \mathbb{R}^p$ has the following dynamics:

$$\begin{aligned} \dot{\eta}_i(t) &= f_i(x_i(t), \eta_i(t)) & t \notin \{t^i_k\}^\infty_{k=1} \\ \eta_i(t^+) &\in G_{ci}(x_i(t), \eta_i(t), x_j(t), \eta_j(t)) & t \in \{t^i_k\}^\infty_{k=1} \end{aligned} \tag{4}$$

where $\eta_i(t^+)$ signifies the value of η_i after an instantaneous change at time t. The map $f_{ci} : \mathbb{R}^n \to \mathbb{R}^p$ governs the continuous evolution of the controller between time instances



(a) The distance between the solutions using continuous control decreases to zero, implying that solutions synchronize asymptotically.



(b) With $t^i_{k+1} - t^i_k \in [0.5, 1]$ for $i \in \{1, 2\}$, $k \in \mathbb{N} \setminus \{0\}$, the intermittent communication prevents the controller from steering the distance between solutions to zero.

Fig. 1. A numerical solution to (1) with a controller given by $u_i = K_i(x_i - x_j)$ with $K_1 = K_2 = [-1, -1]$ under (a) continuous communication and (b) intermittent communication. For each case, solutions $\phi = (\phi_1, \phi_2)$ from $\phi_1(0) = (-2, 0)$ and $\phi_2 = (1, 0)$ are given in planar plots as well as the distance $|x_1 - x_2|$ over time.

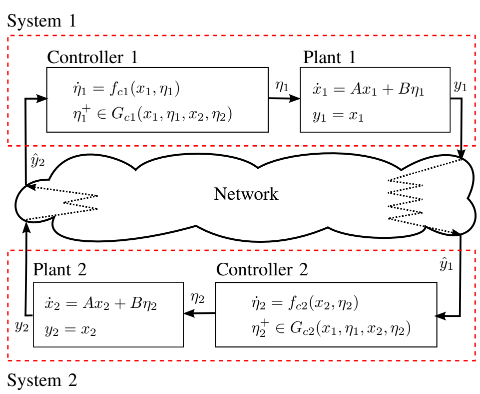


Fig. 2. Two continuous-time systems (plants) with dynamic controllers given by (4) for synchronization over a network with intermittent communication events.

t^i_k while $G_{ci} : \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ is a set-valued map updating the state of the controller when new information arrives. The resulting closed-loop system has the following dynamics:[1]

$$\left.\begin{aligned} \dot{x}_1(t) &= Ax_1(t) + B\eta_1(t) \\ \dot{\eta}_1(t) &= f_{c1}(x_1(t), \eta_1(t)) \end{aligned}\right\} \quad t \notin \{t^1_k\}^\infty_{k=1}$$
$$\left.\begin{aligned} \dot{x}_2(t) &= Ax_2(t) + B\eta_2(t) \\ \dot{\eta}_2(t) &= f_{c2}(x_2(t), \eta_2(t)) \end{aligned}\right\} \quad t \notin \{t^2_k\}^\infty_{k=1}$$
$$\left.\begin{aligned} x_1(t^+) &= x_1(t) \\ \eta_1(t^+) &\in G_{c1}(x_1(t), \eta_1(t), x_2(t), \eta_2(t)) \end{aligned}\right\} \quad t \in \{t^1_k\}^\infty_{k=1}$$
$$\left.\begin{aligned} x_2(t^+) &= x_2(t) \\ \eta_2(t^+) &\in G_{c2}(x_2(t), \eta_2(t), x_1(t), \eta_1(t)) \end{aligned}\right\} \quad t \in \{t^2_k\}^\infty_{k=1} \tag{5}$$

where the functions f_i and g_i are to be designed. Figure 2 shows a block diagram of the closed-loop system, including the complex network. Due to the nonlinearities, nonperiodic arrival of information, and impulsive dynamics, classical control design provides little insight into solving this problem. This motivates us to design the controller in (4) by recasting (5) in a hybrid system framework, precisely the framework in [23]. In the next section, we introduce this framework for hybrid systems and recast the model in (5) into that framework.

[1] When $t \in \{t^1_k\}^\infty_{k=1} \cap \{t^2_k\}^\infty_{k=1}$, then both resets occur.

B. Hybrid Modeling

The model given in (5) consists of a continuous-time plant and a dynamic controller that is allowed to evolve both continuously between transmission time instances (i.e., when t satisfies $t \in [t^i_{k+1}, t^i_k)$) and discretely when information is received (i.e., when $t = t^i_k$, $k \in \mathbb{N} \setminus \{0\}$). Following [24], for each system $i \in \{1, 2\}$, we model these events using a timer variable τ_i that decreases during flows and, once it reaches zero, is reset to any point in $[T_1, T_2]$.

To model this mechanism and the closed-loop system, we follow the hybrid system framework in [23], where a hybrid system is given by using four objects (C, f, D, G) defining its *data:*

- *Flow set*: a set $C \subset \mathbb{R}^n$ specifying the points where (the continuous evolution or) flows are possible.
- *Flow map*: a single-valued map $f : \mathbb{R}^n \to \mathbb{R}^n$ defining the flows.
- *Jumps set*: a set $D \subset \mathbb{R}^n$ specifying the points where (the discrete evolution or) jumps are possible.
- *Jump map*: a set-valued map $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ defining value of the state after the jumps.

A hybrid system with state $x \in \mathbb{R}^n$ is denoted by $\mathcal{H} = (C, f, D, G)$ and can be written in the compact form

$$\mathcal{H} : \qquad x \in \mathbb{R}^n \quad \begin{cases} \dot{x} = f(x) & x \in C \\ x^+ \in G(x) & x \in D. \end{cases} \tag{6}$$

Using this framework, the evolution of the timers τ_i modeling the events is given, for each $i \in \{1, 2\}$, by

$$\begin{aligned} \dot{\tau}_i &= -1 & \tau_i &\in [0, T_2], \\ \tau_i^+ &\in [T_1, T_2] & \tau_i &= 0. \end{aligned} \tag{7}$$

Note that the reset law of the timer state is set valued and it resets to any value within the interval $[T_1, T_2]$. Due to this fact, any sequence of points $\{t^i_k\}^\infty_{k=1}$ that satisies (2) with χ_i given by any probability density function can be modeled.

A model of the closed-loop system, denoted $\mathcal{H}$, has state $x = (\tilde{x}_1, \tilde{x}_2)$ where, for each $i \in \{1, 2\}$, $\tilde{x}_i = (x_i, \eta_i, \tau_i) \in \mathbb{R}^n \times \mathbb{R}^p \times [0, T_2] =: S$. The flow and the jump sets are defined to constrain the evolution of the timers. From the model in (7), the controller dynamics in (4), and the LTI model given by (1), the flow set is defined by constraining the timer states into the interval $[0, T_2]$. Then, we have

$$C := \{x \in S^2 : \tau_1 \in [0, T_2], \tau_2 \in [0, T_2]\} \tag{8}$$

The flow map is given by

$$f(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \end{bmatrix} \tag{9}$$

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where, for each $i \in \{1,2\}$, $f_i(x) = (Ax_i + B\eta_i, f_{ci}(y_1, \eta_i), -1)$. The impulsive events are captured by the jump set D and the jump map G. Since jumps occur when either $\tau_1 = 0$ or $\tau_2 = 0$, the jump set D is defined by the union of the sets D_1 and D_2, that is $D := D_1 \cup D_2$, where

$$D_1 = \{x \in S : \tau_1 = 0\}, \quad D_2 = \{x \in S : \tau_2 = 0\}. \quad (10)$$

Consider the case of $x \in D_1 \setminus D_2$ (i.e., $\tau_1 = 0$ and $\tau_2 > 0$) then a jump is triggered such that $\tilde{x}_1$ is updated via G_1 and $\tilde{x}_2$ is mapped to itself. Likewise, when $x \in D_2 \setminus D_1$, $\tilde{x}_2$ is updated via G_2 and $\tilde{x}_1$ is mapped to itself. Such dynamics are captured by the jump map given by

$$G(x) := \begin{cases} \begin{bmatrix} G_1(x) \\ x_2 \end{bmatrix} & \text{if } x \in D_1 \setminus D_2 \\ \left\{ \begin{bmatrix} x_1 \\ G_2(x) \end{bmatrix}, \begin{bmatrix} G_1(x) \\ x_2 \end{bmatrix} \right\} & \text{if } x \in D_1 \cap D_2 \\ \begin{bmatrix} \tilde{x}_1 \\ G_2(x) \end{bmatrix} & \text{if } x \in D_2 \setminus D_1 \end{cases} \quad (11)$$

where $G_i(x) = (x_i, G_{ci}(Mx_i, \eta_i, Mx_j, \eta_i), [T_1, T_2])$, for each $i, j \in \{1,2\}$, $j \neq i$. If both τ_1 and τ_2 expire simultaneously, then the second piece in the definition of the jump map indicates that either x_1 is reset or x_2 is reset.

A hybrid system $\mathcal{H} = (C, f, D, G)$ is said to satisfy the hybrid basic conditions if

(A1) C and D are closed sets in $\mathbb{R}^n$;
(A2) $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuous on C;
(A3) $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is an outer semicontinuous[2] set-valued mapping, locally bounded on D, and such that $G(z)$ is nonempty for each $z \in D$.

Note that satisfying the hybrid basic conditions implies that $\mathcal{H}$ is well-posed and with uniform asymptotic stability of a set (defined in Section III) automatically gives robustness to small enough perturbations; see [23] for more information. In Section III-C, we consider several classes of perturbations, including perturbations on the measured information, on the state, and the dynamics. For some of those results to hold, we will need the data of $\mathcal{H}$ to satisfy the following conditions.

Lemma 2.1: If, for each $i \in \{1,2\}$, $f_{ci} : \mathbb{R}^n \to \mathbb{R}^p$ is continuous for all $x \in C$ and $G_{ci} : \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ is outer semicontinuous, bounded and nonempty for every $x \in D$, then the hybrid system $\mathcal{H}$ with data given in (8)-(11) *satisfies the hybrid basic conditions.*

A solution ϕ to $\mathcal{H}$ is parametrized by $(t,j) \in \mathbb{R}_{\geq 0} \times \mathbb{N}$, where t denotes ordinary time and j denotes jump time. The domain $\operatorname{dom}\phi \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ is a hybrid time domain if, for every $(T, J) \in \operatorname{dom}\phi$, the set $\operatorname{dom}\phi \cap ([0,T] \times \{0,1,\ldots,J\})$ can be written as $\cup_{j=0}^{J} (I_j \times \{j\})$, where $I_j = [t_j, t_{j+1}]$ for a time sequence $0 = t_0 \leq t_1 \leq \cdots \leq t_J \leq t_{J+1}$. The t_j's with $j > 0$ define the time instants when the state of the hybrid system jumps and j counts the number of jumps.

[2] A set-valued mapping $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is *outer semicontinuous* if its graphs $\{(x,y) : x \in \mathbb{R}^n, y \in G(x)\}$ is closed, see [23, Lemma 5.10]

A solution to $\mathcal{H}$ is called maximal if it cannot be extended, i.e., it is not a truncated version of another solution. It is called complete if its domain is unbounded. A solution is Zeno if it is complete and its domain is bounded in the t direction. A solution is precompact if it is complete and bounded. The set $\mathcal{S}_\mathcal{H}$ contains all maximal solutions to $\mathcal{H}$ and the set $\mathcal{S}_\mathcal{H}(\xi)$ contains all maximal solutions to $\mathcal{H}$ from ξ.

With a hybrid system having the data above, we immediately have the following result.

Lemma 2.2: Given two positive scalars $0 < T_1 \leq T_2$, any maximal solution ϕ to $\mathcal{H}$ in (6) *is such that for every $T > 0$ and $t + j \geq T$, $j \geq \frac{T - T_2}{T_2 + 1}$ with $(t,j) \in \operatorname{dom}\phi$.*

Using the definition of solutions for hybrid systems above, we define uniform global asymptotic synchronization for (6).

Definition 2.3: The hybrid system $\mathcal{H}$ is said to have

- *uniform synchronization* if there exists a class-$\mathcal{K}_\infty$ function α such that any solution $\phi = (\phi_1, \phi_2)$ to $\mathcal{H}$ satisfies $|\phi_1(t,j) - \phi_2(t,j)| \leq \alpha(|\phi_1(0,0) - \phi_2(0,0)|)$ for all $(t,j) \in \operatorname{dom}\phi$;
- *uniform global attractive synchronization* if for each $\gamma > 0$ and $r > 0$ there exists $T > 0$ such that, for any solution $\phi = (\phi_1, \phi_2)$ to $\mathcal{H}$ with $|\phi_1(0,0) - \phi_2(0,0)| \leq r$, $(t,j) \in \operatorname{dom}\phi$ and $t + j \geq T$ imply $|\phi_1(t,j) - \phi_2(t,j)| \leq \gamma$;
- *uniform global asymptotic synchronization* if it has both uniform synchronization and uniform global asymptotic synchronization.

We recast asymptotic synchronization as a set stabilization problem, where we determine the stability of the set $\mathcal{A} := \{x \in S^2 : x_1 = x_2\}$. See results in [23] for more information.

In the next section, we present the main results of this paper. Specifically, we showcase three unique network structures, and design hybrid controllers for each case and study their robustness. The proof of each result is given in the report in [25].

III. Main Results

In the subsequent sections, we consider the following system topologies:

1) A cascade topology with a unidirectional communication network allowing system 1 to transmit information to system 2 via the network modeled as in (7);
2) A feedback topology with a bidirectional communication network with independent transmission events.

Next, we adapt the model in Section II-B to address each network type and provide a solution to the associated synchronization problem.

A. Cascade Topology

In this section, we address the case of a cascade topology of two systems. In this setting, information is transmitted from system 1 to system 2 only, where system 1 is autonomous. Since the network is unidirectional, we use a single timer, denoted by τ, satisfying the dynamics in (7)

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to trigger the update of information. To adapt the hybrid system model in Section II-B to this setting, we remove the timer and the input to system 1. Then, the cascade topology is reduced to a hybrid system with state $x = (x_1, x_2, \eta_2, \tau) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^p \times [0, T_2] := S_1$ and the following data:

$$\begin{aligned} f(z) &:= \left[Ax_1, Ax_2 + B\eta_2, f_{c2}(x_2), -1\right]^\top \\ C &:= \{(x_1, x_2, \eta_2, \tau) \in S_1 : \tau \in [0, T_2]\} \\ G(z) &:= \left[x_1, x_2, G_{c2}(x_1, x_2), [T_1, T_2]\right]^\top \\ D &:= \{(x_1, x_2, \eta_2, \tau) \in S_1 : \tau = 0\}. \end{aligned} \tag{12}$$

with f_{c2} and G_{c2} defining the hybrid controller during flows and jumps, respectively.

We first consider the case of a generic sample-and-hold controller. In this control structure, the controller state does not change during flows, but is updated at jumps. Following [24], we write the system in error coordinates $(\varepsilon, \eta_2, \tau) \in \mathbb{R}^n \times \mathbb{R}^p \times [0, T_2] =: \widetilde{S}_1$, where $\varepsilon = x_1 - x_2$. The resulting hybrid system is denoted as $\mathcal{H}_\varepsilon$ with data $(\widetilde{C}, \widetilde{f}, \widetilde{D}, \widetilde{G})$. During flows, it follows that

$$\begin{aligned} \dot{\varepsilon} &= \dot{x}_1 - \dot{x}_2 = Ax_1 - Ax_2 - B\eta_2 = A\varepsilon - B\eta_2 \\ \dot{\eta}_2 &= 0 \\ \dot{\tau} &= -1, \end{aligned}$$

where, to model the sample and hold operation, we designed f_{c2} as the zero function. Note that since the dynamics of the timer state did not change from that of (7), the flow set is given by $\widetilde{C} := \{(\varepsilon, \eta_2, \tau) \in \widetilde{S}_1 : \tau \in [0, T_2]\}$. If $\tau = 0$, then a jump is triggered, which impulsively updates the state variables according to

$$\begin{aligned} \varepsilon^+ &= x_1^+ - x_2^+ = x_1 - x_2 = \varepsilon \\ \eta_2^+ &\in G_{c2}(x_1, x_2, \eta_2) =: \widetilde{G}_{c2}(\varepsilon, \eta_2) \\ \tau^+ &\in [T_1, T_2] \end{aligned}$$

where we assume that the states x_1 and x_2 in $\widetilde{G}_{c2}$ appear in the form $x_1 - x_2$ only. The jump set is $\widetilde{D} := \{(\varepsilon, \eta_2, \tau) \in \widetilde{S}_1 : \tau = 0\}$. This change of coordinates leads to the hybrid system with state $(\varepsilon, \eta_2, \tau)$ given by

$$\mathcal{H}_\varepsilon = \begin{cases} \left.\begin{aligned} \dot{\varepsilon} &= A\varepsilon - B\eta_2 \\ \dot{\eta}_2 &= 0 \\ \dot{\tau} &= -1 \end{aligned}\right\} \\ (\varepsilon, \eta_2, \tau) \in \{(\varepsilon, \eta_2, \tau) \in \widetilde{S}_1 : \tau \in [0, T_2]\} =: \widetilde{C}, \\ \left.\begin{aligned} \varepsilon^+ &= \varepsilon \\ \eta_2^+ &\in \widetilde{G}_{c2}(\varepsilon, \eta_2) \\ \tau^+ &\in [T_1, T_2] \end{aligned}\right\} \\ (\varepsilon, \eta_2, \tau) \in \{(\varepsilon, \eta_2, \tau) \in \widetilde{S}_1 : \tau = 0\} =: \widetilde{D} \end{cases} \tag{13}$$

from where $\widetilde{f}(z) = (A\varepsilon - B\eta_2, 0, -1)$ for each $(\varepsilon, \eta_2, \tau) \in \widetilde{C}$ and $\widetilde{G}(z) = (\varepsilon, \widetilde{G}_{c2}(\varepsilon, \eta_2, [T_1, T_2])$ for each $(\varepsilon, \eta_2, \tau) \in \widetilde{D}$. Note that this system is independent of the dynamics of the hybrid system given by the data in (12). Due to the change of variables, we study the synchronization set given by

$$\widetilde{\mathcal{A}} := \{(\varepsilon, \eta_2, \tau) \in \widetilde{S}_1 : \varepsilon = 0, \tau \in [0, T_2]\}. \tag{14}$$

In the next sections, we provide conditions for asymptotic stability of the synchronization set in (14) for the case of a generic $\widetilde{G}_{c2}$ map as well as for a specific choice.

1) Controller with generic $\widetilde{G}_{c2}$: In this section, we consider the case of a generic sample-and-hold controller as seen in $\mathcal{H}_\varepsilon$ in (13). This generic controller is assumed to be a set-valued mapping on the error state and the controller state itself. For example, $\widetilde{G}_{c2}$ could implement two strategies: one that would rapidly drive solutions to a region close to $\widetilde{\mathcal{A}}$ and one that accurately steers the state to $\widetilde{\mathcal{A}}$. In this way, we leave the function to be general and give the following conditions on $\widetilde{G}_{c2}$ to guarantee uniform global asymptotic stability of the set $\widetilde{\mathcal{A}}$.[3]

Proposition 3.1: Given two positive scalars $T_1 \leq T_2$, if there exists a positive definite symmetric matrix $P \in \mathbb{R}^{(n+p)\times(n+p)}$ partitioned as $P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^\top & P_{22} \end{bmatrix}$ and a set-valued map $\widetilde{G}_{c2} : \mathbb{R}^n \times \mathbb{R}^p \rightrightarrows \mathbb{R}^p$ such that for every $(\varepsilon, \eta_2, g, \tau)$ satisfying $(\varepsilon, g) \neq 0$, $(\varepsilon, \eta_2) \neq 0$, $(\varepsilon, \eta_2, \tau) \in D$, $g \in \widetilde{G}_{c2}(\varepsilon, \eta_2)$ and $\nu \in [T_1, T_2]$, we have

$$\begin{aligned} &\begin{bmatrix} \varepsilon^\top & g^\top \end{bmatrix} \begin{bmatrix} \widetilde{P}_{11}(\nu) & \widetilde{P}_{12}(\nu) \\ \widetilde{P}_{12}(\nu)^\top & \widetilde{P}_{22}(\nu) \end{bmatrix} \begin{bmatrix} \varepsilon^\top \\ g^\top \end{bmatrix} - \\ &\qquad \begin{bmatrix} \varepsilon & \eta_2 \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^\top & P_{22} \end{bmatrix} \begin{bmatrix} \varepsilon \\ \eta_2 \end{bmatrix} < 0 \end{aligned} \tag{15}$$

where

$$\begin{aligned} \widetilde{P}_{11}(\nu) &= e^{A^\top \nu} P_{11} e^{A\nu} \\ \widetilde{P}_{12}(\nu) &= e^{A^\top \nu} P_{11} \textstyle\int_0^\nu e^{A(\nu - s)} B ds + e^{A^\top \nu} P_{12} \\ \widetilde{P}_{22}(\nu) &= \left(\textstyle\int_0^\nu e^{A(\nu-s)} B ds\right) P_{12} \textstyle\int_0^\nu e^{A(\nu-s)} B ds \\ &\quad + P_{12}^\top \textstyle\int_0^\nu e^{A(\nu-s)} B ds \\ &\quad + \left(\textstyle\int_0^\nu e^{A(\nu-s)} B ds\right) P_{12} + P_{22}, \end{aligned} \tag{16}$$

then the set $\widetilde{\mathcal{A}}$ in (14) is UGAS for $\mathcal{H}_\varepsilon$ in (13), namely, the system uniformly globally asymptotically synchronizes.

In the next section, we consider a specific case of the jump map $\widetilde{G}_{c2}$ given by a proportional static state-feedback sample-and-hold controller and give sufficient conditions for the asymptotic stability of the synchronization set to hold.

2) Controller with $\widetilde{G}_{c2}(\varepsilon, \eta_2) = K\varepsilon$: As a specific case of the general controller above, consider the hybrid sample-and-hold controller with dynamics

$$\begin{aligned} \dot{\eta}_2 &= 0 \qquad (\varepsilon, \eta_2, \tau) \in \{(\varepsilon, \eta_2, \tau) \in \widetilde{S}_1 : \tau \in [0, T_2]\} \\ \eta_2^+ &= K\varepsilon \quad\; (\varepsilon, \eta_2, \tau) \in \{(\varepsilon, \eta_2, \tau) \in \widetilde{S}_1 : \tau = 0\} \end{aligned} \tag{17}$$

where $K \in \mathbb{R}^{p\times n}$ is the controller gain matrix. The hybrid closed-loop system in (13) with the controller in (17) reduces

[3] We use the definition of uniform global asymptotic stability (UGAS) as defined in [23, Definition 3.6].

to

$$\mathcal{H}_\varepsilon = \begin{cases} \left.\begin{array}{ll} \dot{\varepsilon} & = A\varepsilon - B\eta_2 \\ \dot{\eta}_2 & = 0 \\ \dot{\tau} & = -1 \end{array}\right\} \\ (\varepsilon, \eta_2, \tau) \in \{(\varepsilon, \eta_2, \tau) \in \widetilde{S}_1 : \tau \in [0, T_2]\}, \\ \left.\begin{array}{ll} \varepsilon^+ & = \varepsilon \\ \eta_2^+ & = K\varepsilon \\ \tau^+ & \in [T_1, T_2] \end{array}\right\} \\ (\varepsilon, \eta_2, \tau) \in \{(\varepsilon, \eta_2, \tau) \in \widetilde{S}_1 : \tau = 0\}. \end{cases} \tag{18}$$

Following [23, Example 3.21], we partition the state as $z = (z_1, z_2) = (\varepsilon, \eta_2, \tau)$, where $z_1 = (\varepsilon, \eta_2)$ and $z_2 = \tau$. Then, we define the data of $\mathcal{H}_\varepsilon = (\widetilde{C}, \tilde{f}, \widetilde{D}, \widetilde{G})$ as

$$\tilde{f}(z) = \begin{bmatrix} A_f z_1 \\ -1 \end{bmatrix} \qquad \widetilde{G}(z) = \begin{bmatrix} A_g z_1 \\ [T_1, T_2] \end{bmatrix} \tag{19}$$

where

$$A_f = \begin{bmatrix} A & -B \\ 0 & 0 \end{bmatrix} \qquad A_g = \begin{bmatrix} I & 0 \\ K & 0 \end{bmatrix}. \tag{20}$$

Furthermore, the flow and jump sets are $\widetilde{C} := \{(z_1, z_2) \in \widetilde{S}_1 : z_2 \in [0, T_2]\}$ and $\widetilde{D} := \{(z_1, z_2) \in \widetilde{S}_1 : z_2 = 0\}$, respectively. Then, the goal is to determine the stability of the set

$$\widetilde{\mathcal{A}} := \{(z_1, z_2) \in \widetilde{S}_1 : z_1 = 0, z_2 \in [0, T_2]\}. \tag{21}$$

The following result utilizes sufficient condition for stability of hybrid systems to determine the stability of $\mathcal{A}$ for $\mathcal{H}_\varepsilon$, see [23] for more information. Moreover, we can characterize the rate of convergence by upper bounding solutions by a class-$\mathcal{KL}$ function.

Theorem 3.2: Given two positive scalars $T_1 \le T_2$, if there exists a positive definite symmetric matrix $P \in \mathbb{R}^{(n+p)\times(n+p)}$ and a matrix $K \in \mathbb{R}^{p\times n}$ such that

$$A_g^\top e^{A_f^\top \nu} P e^{A_f \nu} A_g - P < 0 \qquad \forall \nu \in [T_1, T_2], \tag{22}$$

then the set $\mathcal{A}$ defined in (21) *is UGAS for the hybrid system* (18)*, namely, the system uniformly globally asymptotically synchronizes. Furthermore, any maximal solution ϕ is bounded by*

$$|\phi(t,j)|_{\mathcal{A}} \le \sqrt{\frac{\overline{c}}{\underline{c}}} e^{\theta j/2} |\phi(0,0)|_{\mathcal{A}} \tag{23}$$

for every $(t,j) \in \operatorname{dom}\phi$, where $\theta = \ln(1 - \beta/\underline{c})$,

$$\beta \le \min_{\tau\in[T_1,T_2]} |A_g^\top e^{A_f^\top \nu} P e^{A_f \nu} A_g - P|,$$

$$\underline{c} = \min_{\tau\in[0,T_2]} \lambda_{\min}(e^{A_f^\top \tau} P e^{A_f \tau}),$$

and

$$\overline{c} = \max_{\tau\in[0,T_2]} \lambda_{\max}(e^{A_f^\top \tau} P e^{A_f \tau}).$$

Remark 3.3: Note that condition (22) is akin to the discrete Lyapunov equation with system matrix $H(\nu) = e^{A_f \nu} A_g$. Furthermore, condition (22) is satisfied if the eigenvalues of $H(\nu)$ are within the unit circle for all $\nu \in [T_1, T_2]$.

Remark 3.4: Due to the form of condition (22), it can be difficult to numerically satisfy. In fact, this condition is not convex in K and P and need to be verified for infinitely many values of ν. In [24], the authors use a polytopic embedding strategy, in which, one needs to find some matrices X_i such that the exponential matrix is an element in the convex hull of the X_i matrices to solve an linear matrix inequality. We present some results similar to this in [25].

Remark 3.5: When full state measurements are not available, but rather, the output $y_i = Mx_i$ is measured, the controller's jump map is given by $\widetilde{G}_{c2}(\varepsilon, \eta_2) = KM\varepsilon$. By following Lemma 3.2, given $T_1 \le T_2$, if there exist K and $P = P^\top > 0$ of appropriate dimensions such that (22) holds with $A_g = \begin{bmatrix} I & 0 \\ KC & 0 \end{bmatrix}$, then $\widetilde{\mathcal{A}}$ is UGAS.

In the next section, we study a bidirectional communication topology with two independent timers triggering the updates. In [25], we also consider the case of a bidirectional communication topology with a single timer, which can be studied using the methods presented in this section.

B. Feedback topology with bidirectional communication having independent clocks and jump map $\widetilde{G}_{ci} = K\varepsilon$

Consider the "synchronization error" hybrid system $\mathcal{H}_\varepsilon$ with state $z = (\tilde{z}_1, \tilde{z}_2) \in \mathbb{R}^{n+p} \times [0, T_2] \times \mathbb{R}^{n+p} \times [0, T_2] =: S^2$, where $\tilde{z}_i = (z_i, \tau_i)$ and $z_i = (\varepsilon_{ij}, \eta_i)$ with ε_{ij} being the error quantity $\varepsilon_{ij} = x_i - x_j$ for each $i, j \in \{1, 2\}$, $i \ne j$. By transferring the states into these coordinates, for each $i \in \{1, 2\}$, the continuous dynamics of each z_i are given by

$$\dot{z}_i = A_f z_i + B_f z_j \tag{24}$$

for all z in $C := \{z \in S : \tau_1 \in [0, T_2], \tau_2 \in [0, T_2]\}$, while at jumps (i.e., $z \in D$), z_i is updated by

$$z_i^+ = \begin{cases} A_g z_i & \text{if } \tau_i = 0 \\ z_i & \text{otherwise} \end{cases} \tag{25}$$

where the system matrices are given by

$$A_f = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}, \; B_f = \begin{bmatrix} 0 & -B \\ 0 & 0 \end{bmatrix}, \; A_g = \begin{bmatrix} I & 0 \\ K & 0 \end{bmatrix}. \tag{26}$$

From the above continuous and discrete dynamics, the hybrid system $\mathcal{H}_\varepsilon$ is defined by the following data:

$$\tilde{f}(z) = (f_1(z), f_2(z)) \tag{27}$$

for each $z \in \widetilde{C} := \{z \in S : \tau_1 \in [0, T_2], \tau_2 \in [0, T_2]\}$ where, for each $i, j \in \{1, 2\}, i \ne j$, the flow map for each system is given by $f_i(z) = (A_f z_i + B_f z_j, -1)$, and the jump map by

$$\widetilde{G}(z) = \begin{cases} \begin{bmatrix} \widetilde{G}_1(z) \\ \tilde{z}_2 \end{bmatrix} & \text{if } z \in \widetilde{D}_1 \setminus \widetilde{D}_2 \\ \begin{bmatrix} \tilde{z}_1 \\ \widetilde{G}_2(z) \end{bmatrix} & \text{if } z \in \widetilde{D}_2 \setminus \widetilde{D}_1 \\ \left\{ \begin{bmatrix} \widetilde{G}_1(z) \\ \tilde{z}_2 \end{bmatrix}, \begin{bmatrix} \tilde{z}_1 \\ \widetilde{G}_2(z) \end{bmatrix} \right\} & \text{if } z \in \widetilde{D}_2 \cap \widetilde{D}_1 \end{cases} \tag{28}$$

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for all $z \in \widetilde{D} := \widetilde{D}_1 \cup \widetilde{D}_2$ with $\widetilde{D}_i := \{z \in S : \tau_i = 0\}$ and $\widetilde{G}_i(z) = (A_g z_i, [T_1, T_2])$ for each $i \in \{1, 2\}$. The main result in this section lays out the conditions on the controller gain K that will drive each error state ε_{12} and ε_{21} to the origin, which, in turn, leads to the controller states also converging to zero. Then, the set to stabilize is given by

$$\widetilde{\mathcal{A}} = \{z \in S : z_i = 0, \tau_i \in [0, T_2], i \in \{1, 2\}\}. \qquad (29)$$

The proof of the following result uses a Lyapunov function based approach show that $\widetilde{\mathcal{A}}$ is uniformly globally asymptotically stable.

Theorem 3.6: Given two positive scalars $T_1 \leq T_2$, if there exist $\epsilon, \sigma, \beta > 0$ such that $\beta < \underline{c}$, and

$$1 - \frac{\beta}{\underline{c}} < \exp\left(-T_2\left(\frac{\rho}{\underline{c}} - \sigma + \frac{1}{\epsilon}\right)\right),$$

a matrix $K \in \mathbb{R}^{n \times p}$, and a positive symmetric matrix $P \in \mathbb{R}^{(n+p)\times(n+p)}$ satisfying

$$e^{\sigma\nu} A_g^\top e^{A_f^\top \nu} P e^{A_f \nu} A_g - P < -\beta I \qquad \forall \nu \in [T_1, T_2] \quad (30)$$

where $\underline{c} = \min_{\tau\in[0,T_2]} \lambda_{\min}(e^{A_f^\top \tau} P e^{A_f \tau})$, and $\rho = \epsilon e^{\sigma T_2} \max_{\tau\in[0,T_2]} \left| B^\top e^{A^\top \tau} P_{11} e^{A\tau} B \right|$, then the set $\widetilde{\mathcal{A}}$ in (29) *is UGAS for the hybrid system with data given by* (27) *and* (28)*, namely, the system uniformly globally asymptotically synchronizes.*

In the next section, we study the robustness of asymptotic synchronization with various perturbations on the systems previously presented in this paper.

C. Robustness properties of the closed-loop systems

1) Robustness to Communication Noise: In a realistic setting, the information transmitted would be subjected to some amount of noise. In this section, we consider system (12) under the effect of measurement noise m_i induced by perturbations in the communication network between system 1 and system 2. Specifically, the j-th system receives the output of the i-th system perturbed by $m_i \in \mathbb{R}^n$, i.e., $y_i = x_i + m_i$. In such a case, the update law in the controller (17) becomes $\eta_2^+ = K\varepsilon + K\overline{m}$, where $\overline{m} = m_1 + m_2$. Note that the larger the controller gain the higher the amplification of the noise. Inspired by [24], we show that the hybrid systems in (18) are input-to-state (ISS) stable with respect to noise measurements m_i. For a notion of ISS for hybrid systems, see [26].

Theorem 3.7: Given two positive scalars $T_1 \leq T_2$, if there exists a symmetric positive definite matrix $P \in \mathbb{R}^{(n+p)\times(n+p)}$ and a matrix $K \in \mathbb{R}^{p\times n}$ satisfying (22)*, then the hybrid system* (18) *with $\eta_2^+ = K\varepsilon + K\overline{m}$, where $\overline{m} = m_1 - m_2$, is ISS with respect to m_2 relative to the set $\widetilde{\mathcal{A}}$ in* (21)*.*

2) Robustness to Generic Perturbations: Due to the fact that each hybrid system presented in this paper satisfies the hybrid basic conditions, we have that $\mathcal{H}$ is well-posed as defined in [23, Definition 6.2]. Thus, the UGAS property of the compact set $\widetilde{\mathcal{A}}$ in (21) for the nominal system $\mathcal{H}$ holds semiglobally for the perturbed system denoted by $\mathcal{H}_\rho$. Following the construction of the perturbed hybrid system in [23, Definition 6.27], we have that given the perturbation function $\rho : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, the ρ-perturbation of $\mathcal{H}$, denoted by $\mathcal{H}_\rho$ with data $(C_\rho, F_\rho, D_\rho, G_\rho)$ given by

$$\begin{aligned} C_\rho &= \{x \in \mathbb{R}^n : (x + \rho(x)\mathbb{B}) \cap C \neq \emptyset\} \\ F_\rho(x) &= \overline{\text{con}} F((x + \rho(x)\mathbb{B}) \cap C) + \rho(x)\mathbb{B} \qquad \forall x \in \mathbb{R}^n \\ D_\rho &= \{x \in \mathbb{R}^n : (x + \rho(x)\mathbb{B}) \cap D \neq \emptyset\} \\ G_\rho(x) &= \{v \in \mathbb{R}^n : v \in g + \rho(g)\mathbb{B}, g \in G((x + \rho(x)\mathbb{B}) \cap D)\} \end{aligned}$$

for each $x \in \mathbb{R}^n$. Then, for each system which has $\widetilde{\mathcal{A}}$ UGAS the following result applies.

Theorem 3.8: (robustness to asymptotic stability) If $\rho : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is continuous and positive on $\mathbb{R}^n \setminus \widetilde{\mathcal{A}}$, then $\mathcal{A}$ is semiglobally practically robustly $\mathcal{KL}$ asymptotically stable, i.e., for every compact set $\mathcal{J} \subset S^2$ and every $\iota > 0$, there exists $\delta \in (0, 1)$ such that every maximal solution ϕ to $\mathcal{H}_{\rho,\delta}$ from $\mathcal{J}$ satisfies $|\phi(t, j)|_{\mathcal{A}} \leq \beta(\phi(0, 0), t + j) + \iota$ for all $(t, j) \in \operatorname{dom} \phi$.

IV. Examples

In this section, we consider several numerical examples. In each of these examples, the timer resets according to a uniform pseudorandom distribution in the interval of T_1 and T_2. The solution to (22) and (30) are obtained following the ideas in [24].

A. Linear oscillator over a feedback topology with bidirectional communication

Now, we revisit the harmonic oscillator example in Section II-A with the same time interval $T_1 = 0.5$ and $T_2 = 1$. From Theorem 3.6, we generate a controller matrix $K = [-0.5, -0.7]$, a positive definite symmetric matrix $P \approx \begin{bmatrix} 20.4 & 2.02 & 3.56 \\ \star & 7.11 & -0.38 \\ \star & \star & 3.56 \end{bmatrix}$, $\beta = 2.6965$, $\epsilon = 3$, $\sigma = 0.4$. Figure 3 shows a numerical solution of this system from initial conditions $\tilde{x}_1(0, 0) = (1, 0, 0, .2)$, $\tilde{x}_2(0, 0) = (-2, 0, 0, .1)$. In this figure, we have (left) an $x_1 - x_2$ planar view of the original states x_1 and x_2 and (right) the Lyapunov function V along solutions (based on the error coordinates in Section III-B), where $V(z) = \sum_{i=1}^2 e^{\sigma\tau_i} z_i^\top e^{A_f^\top \tau_i} P e^{A_f \tau_i} z_i$. Notice that the Lyapunov function increases at points due to the injection terms in (24). Note that the controller designed for this system asymptotically synchronizes the systems without resorting to arbitrarily fast switching speeds, see Figure 1(b).

B. Unstable first order system on the cascade topology

Consider the unstable LTI system with continuous dynamics given by $\dot{x}_i = x_i + \eta_i$ operating on a cascade network topology with $T_1 = 0.1$, $T_2 = 0.5$. By implementing this system and its hybrid controller into the hybrid framework in (12), we can use the results to find an appropriate controller to stabilize the synchronization set. By Theorem 3.2, we find $K = -1.4$ and the positive definite symmetric matrix given by $P \approx \begin{bmatrix} 13.60 & 0.93 \\ \star & 2.35 \end{bmatrix}$. Figure 4 show a numerical

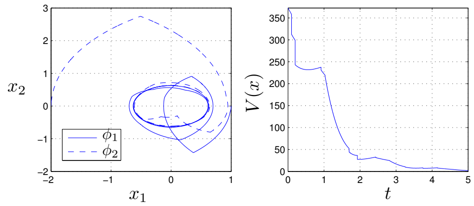

Fig. 3. Solutions to a coupled harmonic oscillator system on the feedback network topology from $\tilde{x}_1(0,0) = (1,0,0,0.2)$ $\tilde{x}_2(0,0) = (-2,0,0,-0.1)$. (right) On the x_1, x_2-planar perspective the solutions converge to a circular orbit about the origin, (left) the Lyapunov function along the solutions decreases to zero, implying that solutions converge to each other asymptotically.

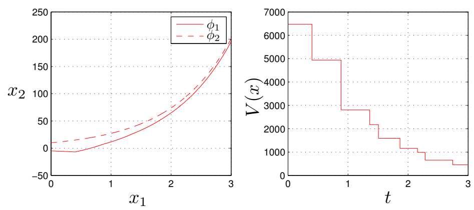

Fig. 4. Solutions of an unstable first order system on the cascade network topology from $x_1(0,0) = (-5,0)$ $x_2(0,0) = (10,0)$ and $\tau(0,0) = 0,4$. While the solutions themselves do not converge to anything, the Lyapunov function along the solutions decreases to zero, implying that solutions converge to each other asymptotically.

simulation of this system from initial conditions $x(0,0) = (-5, 10, 0, .4)$. In this figure, we have (left) a plot of each x_i versus hybrid flow time parameter t. The solutions do not converge to a point; in fact, system 2 converges to system 1 asymptotically. The right figure shows the Lyapunov function V evaluated on solutions, notice that during flows V is constant and during jumps it strictly decreases.

V. Conclusion

In this paper, we modeled two linear time invariant systems connected via two network topologies (cascade and feedback) and have designed a controller to asymptotically synchronize the system states. These properties are established within a solid framework for modeling and analysis of hybrid systems, which has been shown as an accommodating framework for the study of synchronization in other impulsive networks in the literature. Recasting synchronization as a compact set, a Lyapunov function was constructed to certify asymptotic stability of this set, implying that the networked systems asymptotically synchronize. The two networked system studied in this paper provide a solid basis for further development. In fact, some future directions of research include the study of these systems under general graphs with nonlinear dynamics.

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