Robust Distributed Synchronization of Networked Linear Systems with Intermittent Information ⋆

Sean Phillips [a] and Ricardo G. Sanfelice [a]

[a] *Department of Computer Engineering, University of California, Santa Cruz, CA 95064 USA*

Abstract

In this paper, the problem of synchronization of multiple linear time-invariant systems connected over a network with asynchronous and intermittently available communication events is studied. To solve this problem, we propose a controller with hybrid dynamics, namely, the controller utilizes information transmitted to it during discrete communication events and exhibits continuous dynamics between such events. Due to the additional continuous and discrete dynamics inherent to the interconnected networked systems and communication structure, we use a hybrid systems framework to model and analyze the closed-loop system. The problem of synchronization is then recast as a set stabilization problem and, by employing Lyapunov stability tools for hybrid systems, sufficient conditions for asymptotic stability of the synchronization set are provided. Furthermore, we show that the property of synchronization is robust to perturbations. Numerical examples illustrating the main results are included.

Key words: Synchronization; Nonlinear control systems; Communication networks; Distributed control

1 Introduction

1.1 Motivation

The topic of synchronization has gained significant traction in recent years due to the wide range of applications in science and engineering. Synchronization is seen in spiking neurons [1,2], formation control and flocking maneuvers [3,4], distributed sensor networks [5], and satellite constellation formation [6], to name a few. In this paper, we are interested in the topic of synchronization of linear time-invariant systems connected over a general graph where coupling through communication between connected agents occurs only at intermittent time instances. A distributed hybrid controller that only employs such impulsive information to drive the states of the agents in the network to converge to synchronization. The problem comes with many challenges due to the interconnection between each agent being impulsive, which, under the effect of the hybrid controller, results in a hybrid system. Some of the main challenges in designing a control algorithm for synchronization in such a setting include:

- *Asynchronous and heterogenous communication events at unknown times*: the time instances at which each agent receives information are not synchronized and do not necessarily occur periodically. Namely, each agent may receive information from its neighbors at different and unknown time instances. Furthermore, the amount of ordinary time elapsed between consecutive communication events for each agent is not constant and not predefined beforehand; for example, one agent may receive information at a much faster "rate" than others.
- *Instability of nominal dynamics:* each of the systems may not be stable, potentially leading to unbounded trajectories in each system. In particular, the individual dynamics of the agents to be synchronized could be such that their origin is marginally stable or unstable, in which case the state trajectories of the agents need to converge to each other while potentially escaping to infinity.
- *Perturbations in the dynamics, parameters, and measurements:* unknown dynamics in the model makes it difficult to design an algorithm that guarantees exact synchronization. Synchronization algorithms that are

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Email addresses: `seaphill@ucsc.edu` (Sean Phillips), `ricardo@ucsc.edu` (Ricardo G. Sanfelice).

not robust to perturbations on the transmitted information and on the times at which such information arrives could prevent the state trajectories of the agents to converge to nearby values.

1.2 Related Work

The wide applicability of synchronization in science and engineering has promoted a rich set of theoretical results for a variety of class of dynamical systems using a diverse set of tools. The study of convergence and stability of synchronization come through the use of systems theory tools such as Lyapunov functions [7,8], contraction theory [9], and incremental input-to-state stability [10,11]. Results for asymptotic synchronization with continuous coupling between agents exist in both the continuous-time domain and the discrete-time domain; see, e.g., [12,13,14], where the latter is a detailed survey of coordination and consensus for first-order integrator dynamics, in continuous-time and discrete-time. In [12], both for continuous-time and discrete-time interconnected linear time-invariant systems a dynamic control law is shown to guarantee that the solution of each agent converge to that of an homogeneous system with the same dynamics. In [13], the author provides a brief survey on the convergence to synchronization through Lyapunov and set convexity analysis. As pointed out therein, a typical approach to guarantee that the interconnected agents converge to synchronization is to leverage the properties of the graph structure inherent in the connection of multiple agents. Namely, the approach is to use the properties of the graph Laplacian matrix to show that every agent converges to the synchronization manifold; however, as we mention in Remark 3.7, typically only convergence of the solutions to this manifold is discussed and stability is typically left out of the definitions of asymptotic synchronization; see, e.g., [7,12,14].

Synchronization in continuous-time systems where communication coupling occurs at discrete events is an emergent area of study. In [11], the authors study a case of synchronization where agents have nonlinear continuous-time dynamics with continuous coupling and impulsive perturbations. In [15], the authors use Lyapunov-like analysis to derive sufficient conditions for the synchronization of continuously coupled nonlinear systems with impulsive resets on the difference between neighboring agents. Similar to impulsive systems, synchronization in systems where feedback controllers are designed as state-triggered discrete events appeared [16,17]. In [16], a distributed event-triggered control strategy was developed to drive the outputs of the agents in a network to synchronization. Through a Laplacian analysis on solutions of the closed-loop system, an observer-based event policy was developed in [17] for a network of linear time-invariant systems where communication is triggered when the distance between the local state and its estimate is large than a threshold. Using a sample-and-hold self-triggered controller policy, a practical synchronization result was established in [18] for the case of first-order integrator dynamics. To the best of our knowledge, methods for the design of algorithms that guarantee synchronization of multi-agent systems with information arriving at impulsive, asynchronous time instances are not available.

1.3 Problem Formulation, Outline of Proposed Solution, and Contributions

We consider the problem of robustly synchronizing (in terms of both exponential attractivity and stability) $N > 1$ continuous-time agents with linear dynamics (under nominal conditions) from intermittent measurements of functions of their outputs over a network. Namely, we consider the following differential equation modeling the evolution of the state of the i-th agent:

$$\dot{x}_i = Ax_i + Bu_i + \Delta_i(x_i, t) \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$ is the nominal system matrix, $B \in \mathbb{R}^{n\times p}$ is the input matrix, u_i is the control input, $\Delta_i : \mathbb{R}^n \times \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ models unknown and possibly heterogeneous dynamics, and $t \geq 0$ denotes ordinary time. The i-th agent in the network measures its local information y_i and received information from its neighbors y_k at times $t \in \{t^i_s\}^{\infty}_{s=1}$. Moreover, at such event times, the output of each agent is given by

$$y_i = Hx_i + \varphi_i(x_i, t) \tag{2}$$

where H is the output matrix and φ_i is an unknown function modeling communication noise. The event times t^i_s are independently defined for each agent (as the index i denotes); the only restriction imposed on communication times is that they must satisfy

$$\begin{aligned} t^i_{s+1} - t^i_s &\in [T^i_1, T^i_2] \qquad \forall s \in \{1, 2, \dots\} \\ t^i_1 &\leq T^i_2 \end{aligned} \tag{3}$$

where, for each $i \in \mathcal{V}$, the positive scalars T^i_1 and T^i_2 satisfy $T^i_2 \geq T^i_1$ and define the lower and upper bounds on the communication rate, respectively. Namely, these parameters (which are known but may be different for each agent) govern the amount of time allowed to elapse between consecutive communication events. The parameter T^i_2 is often referred to as the *maximum allowable time interval* (MATI).

Motivated by the challenges outlined in Section 1.1, we propose a distributed hybrid controller capable of asymptotically synchronizing the state of each agent over the network, with stability and robustness, by only exchanging information among neighbors at independent communication events t^i_s. In the nominal case, the algorithm proposed here guarantees global exponential stability of the set characterizing synchronization, called the synchronization set, and when projected to the state space of all agents, the synchronization set is the set of points $x = (x_1, x_2, \dots, x_N)$ such that

$$x_1 = x_2 = \cdots = x_N.$$

Moreover, in the presence of small enough general perturbations, the proposed algorithm guarantees that the stability properties are preserved, semiglobally and practically. Under the perturbation effect of measurement noise, we also show that the system is input-to-state stable (in the hybrid sense).

The distributed hybrid controller has state variables which have hybrid dynamics; i.e., the internal states are updated both continuously and, at times, are impulsively updated. In general terms, the continuous dynamics of the controller state are given by a differential equation of the form

$$\dot{\eta}_i = f_{ci}(y_i, \eta_i), \tag{4}$$

when no new information is available, while when new information arrives, the internal states are updated according to

$$\eta_i^+ = \sum_{k=1}^{N} g_{ik} G_{ci}^k(\eta_i, \eta_k, y_i, y_k) \tag{5}$$

where $\mathcal{V} := \{1, 2, \dots, N\}$ defines the set of all agents; g_{ik} models the connection between agents i and k, namely, $g_{ik} = 1$ if the k-th agent can share information to agent i and $g_{ik} = 0$ otherwise; the map f_{ci} defines the continuous evolution of the controller state and the map G_{ci}^k defines the impulsive update law when new information is collected from each connected agent. Then, η_i is injected into the continuous-time dynamics of the i-th agent's input u_i and, at communication events, updates its internal state impulsively. Following the hybrid systems framework in [19], we model the continuous dynamics of each agent, the communication events, and the distributed hybrid controller as a closed-loop hybrid model in Section 3.1.

The main contribution of this work lay on the establishment of sufficient conditions for nominal and robust synchronization over networks with intermittent information availability. In fact, the proposed design conditions guarantee the states of each agent converge to synchronization with an exponential rate when information is only available at, possibly, asynchronous and non-periodic time instance. Precisely, as shown in Section 3.3 through an appropriate choice in coordinates we utilize Lyapunov arguments for hybrid systems to establish sufficient conditions that assure global exponential stability of the synchronization set. An in-depth robustness analysis and design procedure are presented in Section 3.5, wherein we establish several key robustness properties. In part, this is enabled by the proposed hybrid controller which is designed to satisfy certain regularity conditions that, under nominal conditions has uniform global asymptotic stability of the synchronization set, guarantees robustness to small enough perturbations. In Section 3.5.1, we provide results on robustness with respect to perturbations emerging from unmodeled dynamics, skewed clocks, as well as communication noise. In Section 3.5.2, results on robustness in the form of an input-to-state stability (ISS) property with respect to communication noise is provided, for which an explicit ISS bound is given.

In Section 4, we provide numerical simulations to illustrate our results. We consider the case of asynchronous update times where the dynamics of the agents have harmonic oscillator dynamics under different scenarios. Namely, we consider the case of six such systems on a ring graph under nominal conditions as well as subjected to communication noise and packet dropouts. Moreover, we consider such harmonic dynamics on a large-scale system ($N = 100$) representing a small-world network as in [20].

This paper extends our preliminary work in our conference papers [21] about synchronization. In [21], we consider the case of N-dimensional linear time-invariant systems under general graphs where communication events are synchronously triggered throughout the network. This paper not only generalizes these results, but also provides complete proofs, which were not available in [21], includes new results (Theorem 3.9, Proposition 3.14 and Theorem 3.16 are new), and numerous new illustrations via examples in Section 4. We should mention our other recent conference papers [22,23] provide a more in-depth treatment of the consensus problem, and some insight on two interconnected LTI systems under different connectivity structures that are not studied here. Additional details that were not included in this paper due to space constraints are available in [24], in particular, details on the case of synchronous communication, which include two sufficient conditions and modeling techniques, ISS bounds under perturbations, and more examples.

2 Preliminaries

2.1 Basic Notation

Given a matrix A, the set $\mathrm{eig}(A)$ contains all eigenvalues of A and $|A| := \max\{|\lambda|^{\frac{1}{2}} : \ \lambda \in \mathrm{eig}(A^\top A)\}$. Given two vectors $u, v \in \mathbb{R}^n$, $|u| := \sqrt{u^\top u}$ and notation $[u^\top \ v^\top]^\top$ is equivalent to (u, v). Given a function $m : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$, $|m|_\infty := \sup_{t \geq 0} |m(t)|$. $\mathbb{Z}_{\geq 1}$ denotes the set of positive integers, i.e., $\mathbb{Z}_{\geq 1} := \{1, 2, 3, \dots\}$.

$\mathbb{N}$ denotes the set of natural numbers including zero, i.e., $\mathbb{N} := \{0, 1, 2, 3, \dots\}$. Given a symmetric matrix P, $\overline{\lambda}(P) := \max\{\lambda : \lambda \in \text{eig}(P)\}$ and $\underline{\lambda}(P) := \min\{\lambda : \lambda \in \text{eig}(P)\}$. Given matrices A, B with proper dimensions, we define the operator $\text{He}(A, B) := A^\top B + B^\top A$; $A \otimes B$ defines the Kronecker product; $\text{diag}(A, B)$ denotes a 2×2 block matrix with A and B being the diagonal entries; and $A * B$ defines the Khatri-Rao product between A and B. Given $N \in \mathbb{Z}_{\geq 1}$, $I_N \in \mathbb{R}^{N \times N}$ defines the identity matrix and $\mathbf{1}_N$ is the vector of N ones. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a class-$\mathcal{KL}$ function, also written $\beta \in \mathcal{KL}$, if it is nondecreasing in its first argument, nonincreasing in its second argument, $\lim_{r \to 0^+} \beta(r, s) = 0$ for each $s \in \mathbb{R}_{\geq 0}$, and $\lim_{s \to \infty} \beta(r, s) = 0$ for each $r \in \mathbb{R}_{\geq 0}$. Given $s \in \mathbb{R}$, $\lfloor s \rfloor$ denotes the largest integer that is smaller than or equal to s.

2.2 Preliminaries on Graph Theory

A directed graph (digraph) is defined as $\Gamma = (\mathcal{V}, \mathcal{E}, \mathcal{G})$. The set of nodes of the digraph are indexed by the elements of $\mathcal{V} = \{1, 2, \dots, N\}$, and the edges are the pairs in the set $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$. Each edge directly links two nodes, i.e., an edge from i to k, denoted by (i, k), implies that agent i can receive information from agent k. The adjacency matrix of the digraph Γ is denoted by $\mathcal{G} \in \mathbb{R}^{N \times N}$, where its (i, k)-th entry g_{ik} is equal to one if $(i, k) \in \mathcal{E}$ and zero otherwise. A digraph is undirected if $g_{ik} = g_{ki}$ for all $i, k \in \mathcal{V}$. Without loss of generality, we assume that $g_{ii} = 0$ for all $i \in \mathcal{V}$. The in-degree and out-degree of agent i are defined by $d_i^{in} = \sum_{k=1}^N g_{ik}$ and $d_i^{out} = \sum_{k=1}^N g_{ki}$. The in-degree matrix $\mathcal{D}$ is the diagonal matrix with entries $D_{ii} = d_i^{in}$ for all $i \in \mathcal{V}$. The Laplacian matrix of the graph Γ, denoted by $\mathcal{L} \in \mathbb{R}^{N \times N}$, is defined as $\mathcal{L} = \mathcal{D} - \mathcal{G}$. The set of indices corresponding to the neighbors that can send information to the i-th agent is denoted by $\mathcal{N}(i) := \{k \in \mathcal{V} : (i, k) \in \mathcal{E}\}$. A directed graph is said to be *strongly connected* if and only if any two distinct nodes of the graph can be connected via a path that traverses the directed edges of the digraph. It is considered *undirected* if communication between every distinct node is bidirectional, namely, for each edge (i, k) in the edge set $\mathcal{E}$, the edge (k, i) is also in the edge set. Let the digraph be strongly connected and $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N$ be the eigenvalues of $\mathcal{L}$. [1] Then, $\lambda_1 = 0$ is a simple eigenvalue of $\mathcal{L}$ associated with the eigenvector 1_N; $\mathcal{L}$ is positive semi-definite and, therefore, there exists an orthonormal matrix $\Psi \in \mathbb{R}^{N \times N}$ such that $\Psi \mathcal{L} \Psi^\top = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_N)$. The digraph is undirected if and only if the Laplacian is symmetric. Inspired by [26] if the Laplacian is symmetric then we have the following properties. We define $\widetilde{\Psi} = (\psi_2, \psi_3, \dots, \psi_N) \in \mathbb{R}^{N \times N-1}$ with $\psi_i = (\psi_{i1}, \psi_{i2}, \dots, \psi_{iN})$ being the orthonormal eigenvector corresponding to the nonzero eigenvalue λ_i, $i \in \{2, 3, \dots, N\}$, which satisfies $\sum_{k=1}^N \psi_{ik} = 0$. Moreover, $\widetilde{\Psi}$ satisfies the following:

$$\widetilde{\Psi}\widetilde{\Psi}^\top = \frac{1}{N} \begin{bmatrix} N-1 & -1 & \dots & -1 \\ -1 & N-1 & \dots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \dots & N-1 \end{bmatrix} =: U \qquad (6)$$

$\widetilde{\Psi}^\top \widetilde{\Psi} = I$, $U^2 = U$, $\Lambda := \widetilde{\Psi}^\top \mathcal{L} \widetilde{\Psi} = \text{diag}(\lambda_2, \lambda_3, \dots, \lambda_N)$. Note that $\widetilde{\Psi}$ has smaller dimension than Ψ, namely, $\widetilde{\Psi}$ does not contain the eigenvector associated to the zero eigenvalue of the Laplacian.

2.3 Preliminaries on Hybrid Systems

In this paper, a hybrid system $\mathcal{H}$ has data (C, f, D, G) and is defined by

$$\begin{aligned} \dot{\xi} &= f(\xi) \quad \xi \in C, \\ \xi^+ &\in G(\xi) \quad \xi \in D, \end{aligned} \qquad (7)$$

where $\xi \in \mathbb{R}^n$ is the state, f defines the flow map capturing the continuous dynamics and C defines the flow set on which f is effective. The map G defines the jump map and models the discrete behavior, while D defines the jump set, which is the set of points from where jumps are allowed. A solution ϕ to $\mathcal{H}$ is parametrized by $(t, j) \in \mathbb{R}_{\geq 0} \times \mathbb{N}$, where t denotes ordinary time and j denotes jump time. The domain $\text{dom}\, \phi \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ is a hybrid time domain if for every $(T, J) \in \text{dom}\, \phi$, the set $\text{dom}\, \phi \bigcap ([0, T] \times \{0, 1, \dots, J\})$ can be written as the union of sets $\bigcup_{j=0}^J (I_j \times \{j\})$, where $I_j := [t_j, t_{j+1}]$ for a time sequence $0 = t_0 \leq t_1 \leq t_2 \leq \dots \leq t_{J+1}$. The t_j's with $j > 0$ define the time instants when the state of the hybrid system jumps and j counts the number of jumps. A solution to $\mathcal{H}$ is called maximal if it cannot be extended, i.e., it is not a truncated version of another solution. It is called complete if its domain is unbounded. A solution is Zeno if it is complete and its domain is bounded in the t direction. A solution is precompact if it is complete and bounded. The set $\mathcal{S}_\mathcal{H}$ contains all maximal solutions to $\mathcal{H}$, and the set $\mathcal{S}_\mathcal{H}(\xi)$ contains all maximal solutions to $\mathcal{H}$ from ξ.

A hybrid system $\mathcal{H} = (C, f, D, G)$ with data in (7) is said to satisfy the *hybrid basic conditions* if the sets C and D are closed, the function $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuous and the set-valued mapping $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is outer semicontinuous [2] and locally bounded relative to D, and $D \subset \text{dom}\, G$.

[1] See [25] for more information on algebraic graph theory.

[2] A set-valued mapping $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is *outer semicontinuous* if its graph $\{(x, y) : x \in \mathbb{R}^n, y \in G(x)\}$ is closed; see [27, Lemma 5.10].

More information about this hybrid system framework can be found in [27].

3 Robust Global Synchronization with Intermittent Information

3.1 Hybrid Modeling

Consider N agents with dynamics in (1) that are connected via a directed graph. Due to the impulsive nature of the communication structure outlined in (3), we define a decreasing timer to model such a communication scheme. Namely, for each $i \in \mathcal{V}$, let $\tau_i \in [0, T_2^i]$ be a timer state that decreases with respect to continuous time and, upon reaching zero, is reset to a point within the interval $[T_1^i, T_2^i]$ and allows agent i to receive information from its connected agents. Namely, τ_i has the following hybrid dynamics:

$$\begin{aligned} \dot{\tau}_i &= -1 & \tau_i &\in [0, T_2^i], \\ \tau_i^+ &\in [T_1^i, T_2^i] & \tau_i &= 0. \end{aligned} \tag{8}$$

This hybrid system generates any possible sequence of time instances $\{t_s^i\}_{s=1}^{\infty}$ at which events occur and satisfy (3). Note that T_1^i and T_2^i may not be the same for each $i \in \mathcal{V}$, therefore, the interval which communication can occur may be vastly different. [3]

Consider the following definitions of the maps in (4) and (5), which yield the particular hybrid dynamics [4] for η_i therein. The map $f_{ci} : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^p$ is defined as

$$f_{ci}(x_i, \eta_i) = E\eta_i \qquad \forall i \in \mathcal{V} \tag{9}$$

and the map $G_{ci}^k : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \rightrightarrows \mathbb{R}^p$ as

$$\begin{aligned} G_{ci}^k(\eta_i, \eta_k, y_i, y_k) &= K(y_i - y_k) \\ &= KH(x_i - x_k) + K(\varphi_i(x_i) - \varphi_k(x_k)) \end{aligned} \tag{10}$$

for each $i, k \in \mathcal{V}$. The constants E and K define the tuning parameters of the control algorithm. For simplicity, for the remainder of this section, we will assume that $\Delta \equiv 0$ and $\varphi_i \equiv 0$ for all $i \in \mathcal{V}$, and consider the nominal case, i.e., perfect knowledge of the plant dynamics and its output maps. The scenario when these perturbations are nonzero is addressed in Section 3.5. Without such perturbations and with the map (10), the impulsive dynamics of η_i in (5) are given by

$$\eta_i^+ = KH \sum_{k \in \mathcal{N}(i)} (x_i - x_k). \tag{11}$$

For the design of our algorithm for synchronization under intermittent information, we employ the change in coordinates [5]

$$\theta_i = KH \sum_{k \in \mathcal{N}(i)} (x_i - x_k) - \eta_i. \tag{12}$$

which leads to

$$\theta = (\mathcal{L} \otimes KH)x - \eta \tag{13}$$

where $x = (x_1, x_2, \ldots, x_N)$, $\theta = (\theta_1, \theta_2, \ldots, \theta_N)$, $\eta = (\eta_1, \eta_2, \ldots, \eta_N)$, and $\mathcal{L}$ is the Laplacian matrix given by the directed graph Γ of the network. Let $\xi = (z, \tau) \in \mathcal{X} := \mathbb{R}^{(n+p)N} \times \mathcal{T}$ where $z = (x, \theta)$, $\tau = (\tau_1, \tau_2, \ldots, \tau_N)$, and $\mathcal{T} = [0, T_2^1] \times [0, T_2^2] \times \cdots \times [0, T_2^N]$. Then, a hybrid system $\mathcal{H}$ is defined as the collection of all agents with dynamics in (1) and controller states (4) and (5) with data (C, f, D, G) such that, for every $\xi \in C := \mathcal{X}$, we have that

$$\dot{\xi} = (A_f z, -1_N) =: f(\xi). \tag{14}$$

The state matrix A_f is given by

$$A_f = \begin{bmatrix} A_1 & -\widetilde{B} \\ \widetilde{K}A_1 - \widetilde{E}\widetilde{K} & \widetilde{E} - \widetilde{K}\widetilde{B} \end{bmatrix}$$

where $A_1 = I \otimes A + \widetilde{B}\widetilde{K}$, $\widetilde{B} = I \otimes B$, $\widetilde{K} = \mathcal{L} \otimes KH$, and $\widetilde{E} = I \otimes E$. [6] When $\tau_i = 0$, a jump of the i-th agent occurs: the components θ and τ are mapped via $\theta_i^+ = 0$ and $\tau_i^+ \in [T_1^i, T_2^i]$ while x_i remains constant; moreover,

[3] For instance, consider the case of $N = 2$ with $T_1^2 = T_1^1$ and $T_2^2 = 2T_2^1$. At jumps, the timer states τ_1 and τ_2 are reset by $\tau_1 \in [T_1^1, T_2^1]$ when $\tau_1 = 0$ and $\tau_2^+ \in [T_1^1, 2T_2^1]$ when $\tau_2 = 0$; i.e., τ_1 could potentially jump twice as fast as τ_2.

[4] Other choices of the maps f_{ci} and G_{ci}^k might be possible to obtain different dynamics of the variable η_i. Although not pursued in this paper, one can potentially choose sliding mode-like dynamics as in [28].

[5] This change of coordinates was also found useful for the design of observers under intermittent information in [29,30]. Therein, the authors proposed a continuous-time observer design to estimate the state of an LTI plant when its output is available only at intermittent time instances. The observer designed therein uses a memory state (akin to the hybrid controller in this work) that is reset when new measurements are available. Using a similar change of coordinates, sufficient conditions for asymptotic stability of the zero estimation error are derived. These results were extended to the network case in [31].

[6] Through the change of variables in (13), the $z = (x, \theta)$ components of the flow dynamics in (14) are given by $\dot{x} = (I \otimes A)x + (I \otimes B)\eta = (I \otimes A)x + (I \otimes B)(\widetilde{K}x - \theta) = A_1 x - \widetilde{B}\theta$ and $\dot{\theta} = \widetilde{K}\dot{x} - \dot{\eta} = \widetilde{K}(A_1 x - \widetilde{B}\theta) - \widetilde{E}(\widetilde{K}x - \theta)$.

for each $k \in \mathcal{V} \setminus \{i\}$ the state components x_k, θ_k and τ_k are held constant. Specifically, for each $\xi \in D := \cup_{i \in \mathcal{V}} D_i$ where $D_i := \{\xi \in \mathcal{X} : \tau_i = 0\}$, we have that

$$\xi^+ \in G(\xi) := \{G_i(\xi) : \xi \in D_i, i \in \mathcal{V}\} \tag{15}$$

where

$$G_i(\xi) = \begin{bmatrix} x \\ (\theta_1, \theta_2, \ldots, \theta_{i-1}, 0, \theta_{i+1}, \ldots, \theta_N) \\ (\tau_1, \tau_2, \ldots, \tau_{i-1}, [T_1^i, T_2^i], \tau_{i+1}, \ldots, \tau_N) \end{bmatrix}.$$

Lemma 3.1 *Given positive scalars T_1^i and T_2^i such that $T_1^i \leq T_2^i$, the hybrid system $\mathcal{H} = (C, f, D, G)$ with satisfies the hybrid basic conditions.*

Proof By construction, the sets C and D are closed. The flow map f in (14) is continuous. The jump map G is outer semicontinuous since its graph is closed; moreover, it is locally bounded on D. [7] ■

Remark 3.2 *Note that satisfying the hybrid basic conditions implies that the hybrid system $\mathcal{H}$ is well-posed and that asymptotic stability of a compact set as defined in [19, Definition 3.3] is robust to small enough perturbations. See Section 3.5.1 for more information on specific robustness results as a consequence of the hybrid basic conditions.*

3.2 Properties of Maximal Solutions to $\mathcal{H}$

As mentioned in Section 2.3, solutions to a general hybrid system $\mathcal{H}$ can evolve continuously and/or discretely according to the differential and difference equations/inclusions (and the sets where those apply) that describe the hybrid dynamics. The following properties of the domain of maximal solutions are established by exploiting the fact that a timer variable being zero is the only trigger of jumps in the system.

Lemma 3.3 *([32, Lemma 3.5]) Let $0 < T_1^i \leq T_2^i$ be given for all $i \in \mathcal{V}$. Every maximal solution $\phi \in \mathcal{S}_\mathcal{H}$ satisfies the following:*

(1) ϕ is complete; i.e., $\operatorname{dom}\phi$ is unbounded;
(2) for each $(t,j) \in \operatorname{dom}\phi$, $\left(\frac{j}{N} - 1\right)\underline{T} \leq t \leq \frac{j}{N}\overline{T}$, where $\underline{T} := \min_{i\in\mathcal{V}} T_1^i$ and $\overline{T} := \max_{i\in\mathcal{V}} T_2^i$;
(3) for all $j \in \{1, 2, 3, \ldots\}$ such that $(t_{(j+1)N}, (j+1)N), (t_{jN}, jN) \in \operatorname{dom}\phi$, $t_{(j+1)N} - t_{jN} \in [\underline{T}, \overline{T}]$.

[7] The graph of a set-valued mapping $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is defined as gph $G = \{(x, y) : x \in \mathbb{R}^n, y \in G(x)\}$.

3.3 Sufficient Conditions for Synchronization

In this section, we consider the following notion of asymptotic synchronization for hybrid system $\mathcal{H}$.

Definition 3.4 *Consider the hybrid system $\mathcal{H}$ in (7) with state $\xi = (\xi_1, \xi_2, \ldots, \xi_N)$ where, for each $i \in \mathcal{V}$, ξ_i is partitioned as $\xi_i = (p_i, q_i)$, where $p_i \in \mathbb{R}^r$ and $q_i \in \mathbb{R}^{n-r}$ for each $i \in \mathcal{V}$ with integers n, r satisfying $1 \leq r \leq n$. The hybrid system $\mathcal{H}$ is said to have*

- stable synchronization with respect to p *if for every $\varepsilon > 0$, there exists $\delta > 0$ such that every maximal solution $\phi = (\phi_1, \phi_2, \ldots, \phi_N)$ to $\mathcal{H}$, where $\phi_i = (\phi_{i,p}, \phi_{i,q})$ from $|\phi_i(0,0) - \phi_k(0,0)| \leq \delta$ for each $i, k \in \mathcal{V}$ implies $|\phi_{i,p}(t,j) - \phi_{k,p}(t,j)| \leq \varepsilon$ for all $i, k \in \mathcal{V}$ and $(t,j) \in \operatorname{dom}\phi$.*
- globally attractive synchronization with respect to p *if every maximal solution to $\mathcal{H}$ is complete, and for each $i, k \in \mathcal{V}$, $\lim_{\substack{(t,j)\in\operatorname{dom}\phi \\ t+j\to\infty}} |\phi_{i,p}(t,j) - \phi_{k,p}(t,j)| = 0$*
- global asymptotic synchronization with respect to p *if it has both stable synchronization and global attractive synchronization with respect to p.*

Remark 3.5 *If $r = n$, then Definition 3.4 can be considered to be a full-state synchronization notion, while if $r < n$, it can be considered to be a partial state synchronization notion. Note that stable synchronization with respect to p requires solutions ϕ_i for each $i \in \mathcal{V}$ to start close to each other, while only the components $\phi_{i,p}$, $i \in \mathcal{V}$ remain close to each other over their solution domain of definition. Similarly, global attractive synchronization with respect to p only requires that the Euclidean distance between each ϕ_i approaches zero, while the other components are left unconstrained. Also, note that boundedness of the solution is not required.*

It is worth noting that the hybrid systems framework in [27] covers both purely continuous-time and purely discrete-time systems. Namely, continuous-time systems can be modeled in this framework by letting D be empty and G be any arbitrary function, and likewise, discrete-time systems are recovered by letting C be empty and f be any arbitrary function. In the following example, we showcase the notions in Definition 3.4 for the case of continuous-time systems for the synchronization of agents with integrator dynamics.

Example 3.6 *Consider the case of four completely connected agents, i.e., $\mathcal{V} := \{1, 2, 3, 4\}$ and $g_{ik} = 1$ for each $i, k \in \mathcal{V}$ such that $i \neq k$. Each agent has integrator dynamics $\dot{x}_i = u_i$ and is controlled by $u_i = -\gamma \sum_{k=1}^{4}(x_i - x_k)$ where $\gamma > 0$. Integrating the fully interconnected system $\dot{x} = -\gamma\mathcal{L}x$ from $\phi(0,0)$ leads to the complete solution $\phi = (\phi_1, \phi_2, \phi_3, \phi_4)$ with domain $\operatorname{dom}\phi = \mathbb{R}_{\geq 0} \times \{0\}$ given by $\phi_i(t,0) = \frac{1}{4}(\phi_i(0,0)(3\exp(-4\gamma t) + 1) - \sum_{r\in\mathcal{V}\setminus\{i\}} \phi_r(0,0)(\exp(-4\gamma t) - 1))$ for each $i \in \mathcal{V}$ and* *each $t \geq 0$. Stable synchronization can be seen by picking $\delta \in (0, \varepsilon)$ for any given $\varepsilon > 0$. For each $i, k \in \mathcal{V}$ such that $i \neq k$, it follows that $|\phi_i(t,0) - \phi_k(t,0)| = \exp(-4\gamma t)|\phi_i(0,0) - \phi_k(0,0)| \leq \exp(-4\gamma t)\delta < \epsilon$. Moreover, from the derivation above, attractive synchronization is guaranteed, for each $i, k \in \mathcal{V}$, by noting that $\lim_{t\to\infty} |\phi_i(t,0) - \phi_k(t,0)| = 0$. Due to the fact that this system exhibits both stable synchronization and global attractive synchronization, it globally asymptotically synchronizes.* △

Remark 3.7 *There are several notions of synchronization in the literature. A widely used notion of synchronization considers only attractive synchronization in the sense of limits; see, e.g., [12,14,15,7]. Another common notion of synchronization is given as the convergence of all agents to a common solution, namely, that there exists a solution ϕ_s such that, for each $i \in \mathcal{V}$, ϕ_i converges to ϕ_s; see, e.g., [33,17].*

Asymptotic synchronization as defined in Definition 3.4 can be reformulated as a set stability problem. In light of the partial notion of synchronization, our goal is to stabilize the set of points ξ such that each component of x and θ are synchronized. In particular, given a complete solution $\phi = (\phi_x, \phi_\theta, \phi_\tau)$ to the hybrid system $\mathcal{H}$, we want $\lim_{t+j\to\infty} |\phi_{x_i}(t,j) - \phi_{x_k}(t,j)| = 0$ and $\lim_{t+j\to\infty} |\phi_{\theta_i}(t,j) - \phi_{\theta_k}(t,j)| = 0$ for each $i, k \in \mathcal{V}$ where $\phi_x = (\phi_{x_1}, \phi_{x_2}, \dots, \phi_{x_N})$ and $\phi_\theta = (\phi_{\theta_1}, \phi_{\theta_2}, \dots, \phi_{\theta_N})$. To obtain such a property by solving a set stability problem, we define the synchronization set as

$$\begin{aligned} \mathcal{A} = \{\xi = (z, \tau) \in \mathcal{X} : x_1 = x_2 = \dots = x_N, \\ \theta_1 = \theta_2 = \dots = \theta_N\}, \end{aligned} \tag{16}$$

for the hybrid system $\mathcal{H}$ where $\mathcal{X} = \mathbb{R}^{N(n+p)} \times \mathcal{T}$ as defined in Section 3.1.

We consider the following global exponential stability notion of closed sets $\mathcal{A}$ for general hybrid systems $\mathcal{H}$, see [34] for more information.

Definition 3.8 *Let $\mathcal{A} \subset \mathbb{R}^n$ be closed. The set $\mathcal{A}$ is said to be* globally exponentially stable *for the hybrid system $\mathcal{H}$ if every maximal solution to $\mathcal{H}$ is complete and there exist strictly positive scalars κ and r such that for each solution ϕ to $\mathcal{H}$, $|\phi(t,j)|_{\mathcal{A}} \leq \kappa \exp(-r(t+j))|\phi(0,0)|_{\mathcal{A}}$ for all $(t,j) \in \operatorname{dom}\phi$.*

Next, we establish a sufficient condition that guarantees the synchronization property via stability analysis of $\mathcal{A}$ in (16). We establish such a result by using a Lyapunov function. An appropriate choice of V must satisfy $V(\xi) = 0$ for each $\xi \in \mathcal{A}$, while for any $\xi \in \mathcal{X} \setminus \mathcal{A}$, $V(\xi) > 0$. To simplify notation, we introduce the average of the timers of $\mathcal{H}$ given by $\bar{\tau} = \frac{1}{N}\sum_{i=1}^{N} \tau_i$. Inspired by [26], we define the Lyapunov function candidate as

$$V(\xi) = z^\top \bar{\Psi} R(\tau) \bar{\Psi}^\top z, \tag{17}$$

where $\bar{\Psi} = \operatorname{diag}(\widetilde{\Psi} \otimes I_n, \widetilde{\Psi} \otimes I_p)$ with $\widetilde{\Psi}$ defined in Section 2.2, $R(\tau) = \operatorname{diag}(P, Q\exp(\sigma\bar{\tau}))$, $P = \operatorname{diag}(P_2, P_3, \dots, P_N)$, $Q = \operatorname{diag}(Q_2, Q_3, \dots, Q_N)$, $P_i = P_i^\top > 0$, and $Q_i = Q_i^\top > 0$ for each $i \in \{2, 3, \dots, N\}$. The Lyapunov function V in (17) satisfies [27, Definition 3.16], which makes it a suitable Lyapunov function candidate for asymptotic stability of $\mathcal{A}$ in (16). The following result shows that, under certain conditions, for each $\xi \in C$, V decreases during flows, however, at jumps, may have a nonnegative change. Our previous work on distributed estimation with intermittent communication in [32] uses a similar construction of a Lyapunov function. However, such a Lyapunov function decreases during flows and has a non-positive change during jumps. In our setting, to guarantee exponential stability of the synchronization set, we exploit [27, Proposition 3.29], which uses a balancing condition between jumps and flows to guarantee that solutions converge to the desired set.

Theorem 3.9 *Given $0 < T_1^i \leq T_2^i$ for each $i \in \mathcal{V}$ and an undirected connected graph Γ, the set $\mathcal{A}$ is globally exponentially stable for the hybrid system $\mathcal{H}$ with data in (14) and (15) if there exist scalars $\sigma > 0$, $\varepsilon \in (0,1)$, matrices $K \in \mathbb{R}^{n\times p}$ and $E \in \mathbb{R}^{p\times p}$, and positive definite symmetric matrices P_i, Q_i for each $i \in \{2, 3, \dots, N\}$, satisfying*

$$M(\nu) = \begin{bmatrix} \operatorname{He}(P\bar{A}) & -P\bar{B} + \exp(\sigma\nu)(\bar{K}\bar{A} - \bar{E}\bar{K})^\top Q \\ \star & \operatorname{He}(\exp(\sigma\nu)Q(\bar{E} - \bar{K}\bar{B} - \frac{\sigma}{2}I)) \end{bmatrix} < 0 \tag{18}$$

for each $\nu \in [0, \overline{T}]$, where $\bar{A} = I \otimes A + \Lambda \otimes BKH$, $\bar{B} = I \otimes B$, $\bar{E} = I \otimes E$, $\bar{K} = \Lambda \otimes KH$, $\Lambda = diag\lambda_2, \lambda_3, \dots, \lambda_N$ where λ_i are the nonzero eigenvalues of $\mathcal{L}$, and

$$(1-\varepsilon)\underline{T} - \frac{\alpha_2 \sigma \overline{T}}{\beta} > 0. \tag{19}$$

where $\underline{T} := \min_{i\in\mathcal{V}} T_1^i$, $\overline{T} := \max_{i\in\mathcal{V}} T_2^i$,

$$\begin{aligned} \beta &= -\max_{\nu\in[0,\overline{T}]} \bar{\lambda}(M(\nu)) \\ \alpha_2 &= \max\{\overline{\lambda}(P), \overline{\lambda}(Q)\exp(\sigma\overline{T})\}. \end{aligned}$$

Moreover, every $\phi \in \mathcal{S}_{\mathcal{H}}$ satisfies

$$|\phi(t,j)|_{\mathcal{A}} \leq \kappa \exp\left(-r(t+j)\right) |\phi(0,0)|_{\mathcal{A}} \tag{20}$$

for all $(t,j) \in \operatorname{dom}\phi$, where $\kappa = \sqrt{\frac{\alpha_2}{\alpha_1}} \exp\left(\frac{\beta(1-\varepsilon)\underline{T}}{2\alpha_2}\right)$

and $r = \frac{\beta}{2\alpha_2 N}\min\left\{\varepsilon N, (1-\varepsilon)\underline{T} - \frac{\alpha_2\sigma\overline{T}}{\beta}\right\}$, *and* $\alpha_1 = \min\{\underline{\lambda}(P), \underline{\lambda}(Q)\}$.

Proof Consider the Lyapunov function V in (17). Note that, due to the definition of $\bar{\Psi}$, the distance of ξ to the set $\mathcal{A}$ is equivalent to the distance of $\bar{\Psi}^\top z$ to the origin due to the domain of the timer states. More specifically, $|\xi|_{\mathcal{A}}^2 = |\bar{\Psi} z|^2$. Furthermore, from V it follows that

$$\alpha_1|\xi|_{\mathcal{A}}^2 \leq V(\xi) \leq \alpha_2|\xi|_{\mathcal{A}}^2 \tag{21}$$

where α_1 and α_2 are given below (18).[8] During flows, the change in V is given by $\langle\nabla V(\xi), f(\xi)\rangle$ for each $\xi \in C$. To compute such inner product, define $\tilde{R}(\tau) = \mathrm{diag}(0, Q\exp(\sigma\bar{\tau}))$ and note that $\dot{\bar{\tau}} = -1$. Then, it follows that

$$\begin{aligned}\langle\nabla V(\xi), f(\xi)\rangle &= 2z^\top\bar{\Psi}R(\tau)\bar{\Psi}^\top A_f z - \sigma z^\top\bar{\Psi}\tilde{R}(\tau)\bar{\Psi}^\top z \\ &= 2z^\top\bar{\Psi}R(\tau)\bar{\Psi}^\top A_f(I + \bar{\Psi}\bar{\Psi}^\top - \bar{\Psi}\bar{\Psi}^\top)z \\ &\quad - \sigma z^\top\bar{\Psi}\tilde{R}(\tau)\bar{\Psi}^\top z \\ &= 2z^\top\bar{\Psi}R(\tau)\bar{\Psi}^\top A_f\bar{\Psi}\bar{\Psi}^\top z \\ &\quad + 2z^\top\bar{\Psi}R(\tau)\bar{\Psi}^\top A_f(I - \bar{\Psi}\bar{\Psi}^\top)z \\ &\quad - \sigma z^\top\bar{\Psi}\tilde{R}(\tau)\bar{\Psi}^\top z\end{aligned} \tag{22}$$

where we use the property that $\dot{z}^\top\bar{\Psi}R(\tau)\bar{\Psi}^\top z = z^\top\bar{\Psi}R(\tau)\bar{\Psi}^\top\dot{z}$. Recall from Section 2.2 that $\Psi\Psi^\top = U$, $U\mathcal{L} = \mathcal{L}U$ and $\Psi^\top 1 = 0_{N-1\times N}$, which leads to $\bar{\Psi}R(\tau)\bar{\Psi}^\top A_f(I - \bar{\Psi}\bar{\Psi}^\top) = 0$, which reduces (22) to

$$\begin{aligned}\langle\nabla V(\xi), f(\xi)\rangle &= z^\top\bar{\Psi}R(\tau)\bar{\Psi}^\top A_f\bar{\Psi}\bar{\Psi}^\top z \\ &\quad + z^\top\bar{\Psi}\bar{\Psi}^\top A_f^\top\bar{\Psi}R(\tau)\bar{\Psi}^\top z \\ &\quad - \sigma z^\top\bar{\Psi}\tilde{R}(\tau)\bar{\Psi}^\top z \\ &= z^\top\bar{\Psi}(R(\tau)\bar{\Psi}^\top A_f\bar{\Psi} + \bar{\Psi}^\top A_f^\top\bar{\Psi}R(\tau) \\ &\quad - \sigma\tilde{R}(\tau))\bar{\Psi}^\top z.\end{aligned} \tag{23}$$

Due to the definition of $\widetilde{\Psi}\mathcal{L}\widetilde{\Psi}^\top = \Lambda = \mathrm{diag}(\lambda_2, \lambda_3, \ldots, \lambda_N)$, we have that

$$\bar{\Psi}^\top A_f^\top\bar{\Psi} = \begin{bmatrix}\bar{A} & -\bar{B} \\ \bar{K}\bar{A} - \bar{E}\bar{K} & \bar{E} - \bar{K}\bar{B}\end{bmatrix} =: \bar{A}_f$$

where $\bar{A} = I\otimes A + \Lambda\otimes BKH$, $\bar{B} = I\otimes B$, $\bar{E} = I\otimes E$, and $\bar{K} = \Lambda\otimes KH$. Therefore, from (23), it follows that

$$\begin{aligned}\langle\nabla V(\xi), f(\xi)\rangle &= z^\top\bar{\Psi}(R(\tau)\bar{A}_f + \bar{A}_f^\top R(\tau) - \sigma\tilde{R}(\tau))\bar{\Psi}^\top z \\ &= z^\top\bar{\Psi}M(\tau)\bar{\Psi}^\top z\end{aligned}$$

where M is defined in (18). From (18) it follows that

$$\begin{aligned}\langle\nabla V(\xi), f(\xi)\rangle &= z^\top\bar{\Psi}M(\tau)\bar{\Psi}^\top z \\ &\leq -\beta|\xi|_{\mathcal{A}}^2 \leq -\frac{\beta}{\alpha_2}V(\xi)\end{aligned} \tag{24}$$

where $\beta = -\max_{\nu\in\mathcal{T}}\overline{\lambda}(M(\nu))$ and α_2 is defined in (19).

Next, we consider the case $\xi \in D$ and $g \in G(\xi)$. In particular, if there exists at least one component of τ, say, the i-th component, such that $\tau_i = 0$. From the definition of G in (15), x is updated by its identity, $\theta_i^+ = 0$ and $\tau_i^+ \in [T_1^i, T_2^i]$. Moreover, for each $k \in \mathcal{V}\setminus\{i\}$, the k-th component of θ is updated by its identity, i.e., $\theta_k^+ = \theta_k$. Therefore, it follows that during jumps we have that $(\theta^+)^\top\theta^+ \leq \theta^\top\theta$ due to the i-th component being updated to zero when $\tau_i = 0$. Likewise, after the jump of the i-th timer τ_i, we have that τ_i^+ is reset to a point $\nu \in [T_1^i, T_2^i]$. It follows that $\exp(\sigma\bar{\tau}^+) = \exp(\sigma\bar{\tau})\exp(\sigma\frac{\nu}{N})$. Then, the function V after a jump is given by

$$V(g) \leq \exp\left(\sigma\frac{\overline{T}}{N}\right)V(\xi). \tag{25}$$

Note that the quantity $\exp(\sigma\frac{\overline{T}}{N}) - 1$ may be positive.

Next, we evaluate V over a solution to ensure that the distance of the solution ϕ to the set $\mathcal{A}$ converges to zero in the limit as $t + j$ approaches infinity. Pick $\phi \in \mathcal{S}_{\mathcal{H}}$ and any $(t, j) \in \mathrm{dom}\,\phi$. Let $0 = t_0 \leq t_1 \leq t_2 \leq \cdots \leq t_{j+1} \leq t$ satisfy $\mathrm{dom}\,\phi\bigcup([0, t_{j+1}]\times\{0, 1, 2, \ldots, j\}) = \bigcup_{s=0}^{j}([t_s, t_{s+1}]\times\{s\})$ for each $s \in \{0, 1, 2, \ldots, j\}$ and almost all $r \in [t_s, t_{s+1}]$, $\phi(r, s) \in C$. Then, (24) implies that, for each $s \in \{0, 1, 2, \ldots, j\}$ and for almost all $r \in [t_s, t_{s+1}]$,

$$\frac{d}{dr}V(\phi(r, s)) \leq -\frac{\beta}{\alpha_2}V(\phi(r, s)). \tag{26}$$

Integrating both sides of this inequality yields

$$V(\phi(t_{s+1}, s)) \leq \exp\left(-\frac{\beta}{\alpha_2}(t_{s+1} - t_s)\right)V(\phi(t_s, s)) \tag{27}$$

for each $s \in \{0, 1, \ldots, j\}$. Similarly, for each $s \in \{1, 2, \ldots, j\}$, $\phi(t_s, s-1) \in D$, and using (25), we get

$$V(\phi(t_s, s)) \leq \exp\left(\sigma\frac{\overline{T}}{N}\right)V(\phi(t_s, s-1)). \tag{28}$$

It follows, from the previous two inequalities, for each

[8] Note that $\overline{\lambda}(\cdot)$ and $\underline{\lambda}(\cdot)$ are the maximum and minimum eigenvalues, respectively.

$(t, j) \in \operatorname{dom}\phi$,

$$V(\phi(t,j)) \leq \exp\left(-\frac{\beta}{\alpha_2}t + \sigma\frac{\overline{T}}{N}j\right)V(\phi(0,0)) \qquad (29)$$

By virtue of (21) and Lemma 3.3, it follows that (29) becomes

$$|\phi(t,j)|_{\mathcal{A}} \leq \sqrt{\frac{\alpha_2}{\alpha_1}}\exp\left(\frac{\beta(1-\varepsilon)\underline{T}}{2\alpha_2}\right)\exp\left(-\frac{\beta\varepsilon}{2\alpha_2}t + \left(\frac{\sigma\overline{T}}{2N} - \frac{\beta(1-\varepsilon)\underline{T}}{2\alpha_2 N}\right)j\right)|\phi(0,0)|_{\mathcal{A}}$$

where we used the property that there exists $\varepsilon \in (0,1)$ such that $t = \varepsilon t + (1-\varepsilon)t \geq \varepsilon t + (1-\varepsilon)\left(\frac{j}{N} - 1\right)\underline{T}$. Moreover, from (19), and due to every maximal solution to $\mathcal{H}$ being complete, it follows that the bound on $|\phi(t,j)|_{\mathcal{A}}$ implies that $\mathcal{A}$ is globally exponentially stable for the hybrid system $\mathcal{H}$. ■

Remark 3.10 *The matrix inequality in* (18) *comes from the asymptotic stability analysis in the proposed new coordinates* $\xi = (x, \theta, \tau)$, *namely, the analysis during flows; see* (24). *This approach introduces some conservativeness as the reset of* θ_i *to zero when* $\tau_i = 0$ *is not being exploited. This is due to the multiplication of* θ *by* $\widetilde{\Psi} \otimes I_p$ *in* V. *In fact, it is not straightforward to ensure a nonpositive change in* V *during jumps. If such a change could be guaranteed, then the conditions in Theorem 3.9 could be relaxed. Though it exists due to converse theorems, at this time we do not have a Lyapunov function that satisfies the decreasing properties on both jumps and flows.*

Note that the matrix in (1) must be satisfied for an infinite number of points, i.e., $\nu \in [0, \overline{T}]$. Moreover, it can be noted that (18) may be a large matrix in general, which could make finding feasible solutions difficult. It turns out that (18) can be decomposed into $N-1$ matrices due to the fact that each block in the matrix is block diagonal. This leads to the following result.

Proposition 3.11 *Let* $0 < T_1^i \leq T_2^i$ *be given for all* $i \in \mathcal{V}$. *Inequality* (18) *holds if there exist a scalar* $\sigma > 0$ *and matrices* $P_i = P_i^\top > 0$ *and* $Q_i = Q_i^\top > 0$ *for each* $i \in \{2, 3, \ldots, N\}$ *satisfying* $\overline{M}_i(0) < 0$ *and* $\overline{M}_i(\overline{T}) < 0$ *where*

$$\overline{M}_i(\nu) := \begin{bmatrix} \mathrm{He}(P\bar{A}_i) & -PB + \exp(\sigma\nu)(\bar{K}_i\bar{A}_i - E\bar{K}_i)Q_i \\ \star & \mathrm{He}(\exp(\sigma\nu)Q_i(E - \bar{K}_iB - \frac{\sigma}{2}I)) \end{bmatrix} \qquad (30)$$

for each $\lambda_i \in \lambda(\mathcal{L}) \setminus \{0\}$, *where* $\bar{A}_i = A + \lambda_i BKH$ *and* $K_i = \lambda_i KH$.

Proof By definitions of P, Q, Λ and the He operator, the matrix in (18) is a block diagonal matrix. Therefore, we can rearrange M in (18) as the diagonal of $N-1$ sub-matrices $\overline{M}(\nu) = \mathrm{diag}(\overline{M}_2(\nu), \overline{M}_3(\nu), \ldots, \overline{M}_N(\nu))$ where, for each $i \in \{2, 3, \ldots, N\}$, M_i is given by (30). Given $\nu \in [0, \overline{T}]$ and $\sigma > 0$, define the function $r : [0, \overline{T}] \to [0, 1]$ as $r(\nu) = \frac{\exp(\sigma\nu) - \exp(\sigma\overline{T})}{1 - \exp(\sigma\overline{T})}$ for each $\nu \in [0, \overline{T}]$. Then, it can be verified that for any $\nu \in [0, \overline{T}]$ $\exp(\sigma\nu) = r(\nu) + (1 - r(\nu))\exp(\sigma\overline{T})$. Therefore, for each $\nu \in [0, \overline{T}]$, the matrix $\overline{M}_i$ in (30) can be rewritten as $\overline{M}_i(\nu) = r(\nu)\overline{M}_i(0) + (1 - r(\nu))\overline{M}_i(\overline{T})$. By assumption $\overline{M}_i(0) < 0$ and $\overline{M}_i(\overline{T}) < 0$, for each $i \in \{2, \ldots, N\}$, and hence (18) holds for each $\nu \in [0, \overline{T}]$. ■

Remark 3.12 *Note that conditions* $\overline{M}_i(0) < 0$ *and* $\overline{M}_i(\overline{T}) < 0$ *are nonconvex in* P, Q, K, E, *and* σ. *At this time, there is no clear way to reduce the matrices in the conditions into a convex form. In fact, the matrices are bilinear in these variables; therefore, to solve* (30) *one should use a BMI solver such as YALMIP and BMILAB.*

Remark 3.13 *For the case of synchronous communication, a single timer* $\tau \in [0, T_2]$ *can be used to trigger the communication between all agents. Then, through the change of coordinates in* (12) *and following the approach in the proof of Theorem 3.9, it can be shown that if parameters, gains, and matrices exist such that* (18) *is satisfied for all* $\tau \in [0, T_2]$ *then the resulting hybrid system with a single timer has the corresponding synchronization set exponentially stable. We provide more details in a technical report found at [24].*

3.4 Time to Synchronize

Due to its properties along solutions shown in Theorem 3.9, the proposed Lyapunov function can be further exploited to provide a bound on the time to converge to a neighborhood about the synchronization set $\mathcal{A}$. As expected, this time depends on the initial distance to the set $\mathcal{A}$ and the parameters of the hybrid system.

Proposition 3.14 *Given* $0 < T_1^i \leq T_2^i$ *for each* $i \in \mathcal{V}$ *and an undirected connected graph* Γ, *if there exist scalars* $\sigma > 0$ *and* $\varepsilon \in (0,1)$, *matrices* $K \in \mathbb{R}^{n \times p}$ *and* $E \in \mathbb{R}^{p \times p}$, *and positive definite symmetric matrices* P_i, Q_i *for each* $i \in \{2, 3, \ldots, N\}$, (18) *and* (19), *then for each* $c_0 > c_1 > 0$ *every maximal solution* ϕ *to* $\mathcal{H}$ *with initial condition*[9] $\phi(0,0) \in \mathcal{X} \cap L_V(c_0)$ *is such that* $\phi(t,j) \in L_V(c_1)$ *for each* $(t,j) \in \operatorname{dom}\phi$, $t + j \geq \bar{r}$, *where* $\bar{r} = (\frac{N}{\underline{T}} + 1)\Omega + 1$, $\Omega = \frac{\ln\left(\frac{c_1}{c_0}\right) - \sigma\frac{\overline{T}}{N}}{\frac{-\beta}{\alpha_2} + \frac{\sigma\overline{T}}{\underline{T}}}$, *and* $\underline{T}$, $\overline{T}$, β *and* α_2 *are given below* (19).

[9] A sublevel set of V, denoted as $L_V(\mu)$, is given by $L_V(\mu) := \{x \in \mathcal{X} : V(x) \leq \mu\}$.

Proof Let $\phi_0 = \phi(0,0)$ and pick a maximal solution $\phi \in \mathcal{S}_{\mathcal{H}}(\phi_0)$. From the proof of Theorem 3.9, we have that, for each $(t,j) \in \operatorname{dom}\phi$, (29) holds. Namely, for each $(t,j) \in \operatorname{dom}\phi$, V satisfies $V(\phi(t,j)) \leq \exp\left(-\frac{\beta}{\alpha_2}t + \sigma\frac{\overline{T}}{N}j\right)V(\phi_0)$. We want to find $(T,J) \in \operatorname{dom}\phi$ such that $V(\phi(T,J)) \leq c_1$ when $\phi(0,0) \in L_V(c_0)$. Considering the worst case for $V(\phi_0)$, it follows that $c_1 \leq \exp\left(-\frac{\beta}{\alpha_2}T + \sigma\frac{\overline{T}}{N}J\right)c_0$ which implies that $\ln\left(\frac{c_1}{c_0}\right) \leq -\frac{\beta}{\alpha_2}T + \sigma\frac{\overline{T}}{N}J$. Then, from Lemma 3.3, we have that for $(T,J) \in \operatorname{dom}\phi$, it follows that $J \leq N\left(\frac{T}{\underline{T}}+1\right)$ which implies that $T \leq \Omega$ where $\Omega = \frac{\ln\left(\frac{c_1}{c_0}\right) - \sigma\frac{\overline{T}}{N}}{\frac{-\beta}{\alpha_2} + \frac{\sigma\overline{T}}{\underline{T}}}$. Then, after $t + j \geq T + J$, the solution is at least c_1 close to the set $\mathcal{A}$. Defining $\bar{r} = T + J$, we have that $\bar{r} = \left(\frac{N}{\underline{T}}+1\right)\Omega + 1$. ■

3.5 Robustness of Synchronization

In this section, we consider the effect of general perturbations and unmodeled dynamics on the agents in the network. In such a setting, the perturbed model of each agent is given in (1) and the output generated by each agent is given by (2), where the functions $\Delta_i : \mathbb{R}^n \times \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ and $\varphi_i : \mathbb{R}^n \to \mathbb{R}^p$ are unknown functions that may capture the unmodeled dynamics, as well as, the disturbances and communication noise, respectively. In particular, due to the disturbances on the output, the values of y_k transmitted to agent i at communication times $t = t_s$ from agent k (where $k \in \mathcal{N}(i)$) may be affected by some communication channel noise, specifically, $y_k(t_s) = Hx_k(t_s) + \varphi_k(x_k(t_s), t_s)$.

Adding the perturbations to (9) and (10), we have that the continuous dynamics of the distributed controllers do not change, but the discrete dynamics become

$$\eta_i^+ = KH\sum_{k\in\mathcal{N}(i)}(x_i - x_k) + K\tilde{\varphi}_i(x,t) \qquad (31)$$

where $\tilde{\varphi}_i(x,t) = \sum_{k\in\mathcal{N}(i)}(\varphi_i(x_i,t) - \varphi_k(x_k,t))$. For simplicity, hence forth we will drop the arguments of some of the perturbations. We will consider the model in the θ coordinates in Section 3.1 for the study of robustness. Then, following the definition of θ_i in (12), the resulting perturbed hybrid system $\widetilde{\mathcal{H}}$ has data $(C, \widetilde{f}, D, \widetilde{G})$ and state $\xi = (z,\tau) \in \mathcal{X}$, $z = (x,\theta)$. The perturbed data is given by

$$\widetilde{f}(\xi) = f(\xi) + (\Delta(x,t), \widetilde{K}\Delta(x,t), 0) \quad \forall \xi \in C \qquad (32)$$

where $\Delta(t,x) = (\Delta_1(x_1,t), \Delta_2(x_1,t), \ldots, \Delta_N(x_1,t))$ and $\widetilde{K} = (\mathcal{L} \otimes KH)$. Moreover, when $\xi \in D$,

$$\widetilde{G}(\xi,\varphi) := \{\widetilde{G}_i(\xi,\delta) : \xi \in \widetilde{D}_i, i \in \mathcal{V}\} \qquad \forall \xi \in D \qquad (33)$$

and

$$\widetilde{G}(\xi,\varphi) := \begin{bmatrix} x \\ (\theta_1, \theta_2, \ldots, \theta_{i-1}, -K\tilde{\varphi}_i, \theta_{i+1}, \ldots, \theta_N) \\ (\tau_1, \tau_2, \ldots, \tau_{i-1}, [T_1^i, T_2^i], \tau_{i+1}, \ldots, \tau_N) \end{bmatrix}. \qquad (34)$$

In the following sections, we will discuss the robustness of the hybrid system $\widetilde{\mathcal{H}}$ to different classes of perturbations. In particular, we will discuss its robustness to general perturbations on compact set via the hybrid basic conditions and input-to-state stability relative to the synchronization set $\mathcal{A}$.

3.5.1 General Robustness on Compact Sets

In this section, we focus on the generic robustness property to small perturbations. To apply standard robustness results for hybrid systems, the set that is asymptotically stable must be compact. Note that the set $\mathcal{A}$ given by (16) is unbounded: the points $x_1 = x_2 = \cdots = x_N$ and $\theta_1 = \theta_2 = \cdots = \theta_N$ can be any value in $\mathbb{R}^n$ and $\mathbb{R}^p$, respectively. Therefore, we restrict the state space to the compact set $S \times \mathcal{T}$. While this set restricts the state space of the hybrid system, it can easily be considered to be arbitrarily large. The price to pay is that, due to the fact that the state space is now bounded, it is not guaranteed that maximal solutions to the hybrid system are complete. We consider the hybrid system $\widetilde{\mathcal{H}} = (C, \widetilde{f}, D, \widetilde{G})$ as in Section 3.1 with flow and jumps sets given by $\widetilde{C} = C \cap (S \times \mathcal{T})$ and $\widetilde{D} = D \cap (S \times \mathcal{T})$ where $S \subset \mathbb{R}^{N(n+p)}$ is compact. Moreover, the set of interest is given by $\widetilde{\mathcal{A}} = \mathcal{A} \cap (S \times \mathcal{T})$. We have the following result.

Theorem 3.15 *Let $0 < T_1^i \leq T_2^i$ be given for all $i \in \mathcal{V}$. Suppose that the hybrid system satisfies the conditions in Theorem 3.9 for the unperturbed hybrid system $\mathcal{H}$ with data in* (14) *and* (15)*. Then, there exists $\beta \in \mathcal{KL}$ such that, for every compact set $S \subset \mathbb{R}^{N(n+p)}$ and $\varepsilon > 0$, there exists $\rho^\star \geq 0$ such that if $\max\{\bar{\Delta}, \bar{\varphi}\} \leq \rho^\star$ where $\bar{\Delta} = \sup_{(x,t)\in\mathcal{X}\cap(S\times\mathcal{T})\times\mathbb{R}_{\geq 0}} |\Delta(x,t)|$ and $\bar{\varphi} = \sup_{(x,t)\in\mathcal{X}\cap(S\times\mathcal{T})\times\mathbb{R}_{\geq 0}} |\widetilde{\varphi}(x,t)|$ then, every $\phi \in \mathcal{S}_{\widetilde{\mathcal{H}}}(S \times \mathcal{T})$ satisfies $|\phi(t,j)|_{\widetilde{\mathcal{A}}} \leq \beta(|\phi(0,0)|_{\widetilde{\mathcal{A}}}, t+j) + \varepsilon$ for all $(t,j) \in \operatorname{dom}\phi$.*

Proof Consider the hybrid system $\widetilde{\mathcal{H}}$ and a continuous function $\rho : \mathbb{R}^{nN} \times \mathbb{R}^{pN} \times \mathcal{T} \to \mathbb{R}_{\geq 0}$, the ρ-perturbation of $\widetilde{\mathcal{H}}$, denoted $\widetilde{\mathcal{H}}_\rho$, is the hybrid system

$$\begin{cases} \xi \in \widetilde{C}_\rho & \dot{\xi} \in F_\rho(\xi) \\ \xi \in \widetilde{D}_\rho & \xi^+ \in G_\rho(\xi) \end{cases} \tag{35}$$

where

$$\begin{aligned} \widetilde{C}_\rho &= \{\xi \in \widetilde{C} \cup \widetilde{D} : (\xi + \rho(\xi)\mathbb{B}) \cap \widetilde{C} \neq 0\} \\ F_\rho(\xi) &= \overline{\text{con}} f((\xi + \rho(\xi)\mathbb{B}) \cap C = +\rho(\xi)\mathbb{B} \quad \forall \xi \in \widetilde{C} \cap \widetilde{D} \\ \widetilde{D}_\rho &= \{\xi \in \widetilde{C} \cup \widetilde{D} : (\xi + \rho(\xi)\mathbb{B}) \cap \widetilde{D} \neq 0\} \\ G_\rho(\xi) &= \{v \in \widetilde{C} \cap \widetilde{D} : v \in g + \rho(g)\mathbb{B}, g \in G(\xi + \rho(\xi)) \cap \widetilde{D}\} \\ &\qquad \forall \xi \in \widetilde{C} \cap \widetilde{D} \end{aligned}$$

Since the set $\mathcal{A}$ is GES for $\mathcal{H}$, it is also UGAS for $\mathcal{H}$. Since ρ is continuous and $\mathcal{H}$ satisfies the hybrid basic conditions, by [27, Theorem 6.8], $\widetilde{\mathcal{H}}_\rho$ is nominally well-posed and, moreover, by [27, Proposition 6.28] is well-posed. Then, [27, Theorem 7.20] implies that $\mathcal{A}$ is semiglobally practically robustly $\mathcal{KL}$ pre-asymptotically stable for $\widetilde{\mathcal{H}}$. Namely, for every compact set $S \times \mathcal{T} \subset \mathbb{R}^{N(n+p)} \times \mathcal{T}$ and every $\varepsilon > 0$, there exists $\widetilde{\rho} \in (0,1)$ such that every maximal solution ϕ to $\mathcal{H}_{\widetilde{\rho}\rho}$ from $S \times \mathcal{T}$ satisfies $|\phi(t,j)|_{\mathcal{A}\cap(S\times\mathcal{T})} \leq \beta(|\phi(0,0)|_{\mathcal{A}\cap(S\times\mathcal{T})}, t+j) + \varepsilon$ for all $(t,j) \in \operatorname{dom}\phi$. Then, the result follows by picking $\rho^* > 0$ such that $\max\{1, |\widetilde{K}|, |K|\}\rho^* \leq \widetilde{\rho}$ and relating solutions to $\widetilde{\mathcal{H}}$ and solutions to $\mathcal{H}_{\widetilde{\rho}\rho}$. ■

3.5.2 Robustness to Communication Noise

In this section, we consider the hybrid system $\mathcal{H}$ in Section 3.5 when communication noise is present. Namely, φ_i reduces to a function $m_i(t) = \widetilde{\varphi}_i(x_i, t)$ for all $t \in \mathbb{R}_{\geq 0}$ and $i \in \mathcal{V}$. We have the following result.

Theorem 3.16 *Given $0 < T_1^i \leq T_2^i$ for each $i \in \mathcal{V}$ and an undirected connected graph Γ, if there exist scalars $\sigma > 0, \varepsilon \in (0,1)$ and matrices $K \in \mathbb{R}^{n\times p}$ and $H \in \mathbb{R}^{p\times p}$, and positive definite symmetric matrices P_i, Q_i for each $i \in \{2,3,\ldots,N\}$, satisfying* (18) *for each $\nu \in [0,\overline{T}]$ and* (19) *holds, then the set $\mathcal{A}$ is input-to-state stable for the hybrid system $\widetilde{\mathcal{H}}$ in* (32) *and* (33) *with respect to communication noise $m = (m_1, m_2, \ldots, m_N)$. More specifically, for each $\phi \in \mathcal{S}_{\widetilde{\mathcal{H}}}$ and for any $(t,j) \in \operatorname{dom}\phi$,*

$$|\phi(t,j)|_{\mathcal{A}} \leq \max\{\kappa \exp(-r(t+j))\,|\phi(0,0)|_{\mathcal{A}}, \gamma_m |m|_\infty\} \tag{36}$$

where $\underline{T}$, $\overline{T}$, α_2, and β are given below (19) *and κ, r, α_1, are given below* (20), *$b = \exp(\sigma\overline{T}/N)\overline{\lambda}(Q)$, $\gamma_m = NS\sqrt{\frac{\alpha_1}{\alpha_2}}\exp(\sigma\overline{T})b|K|^2$ and $S = \frac{\exp(-\epsilon)}{\exp(-\epsilon)-1}$ where $\epsilon \in \left(0, \frac{\alpha_2\sigma\overline{T}}{\beta} - (1-\varepsilon)\underline{T}\right)$.*

Proof Consider the Lyapunov function candidate $V : \mathcal{X} \to \mathbb{R}_{\geq 0}$ given by (17). It follows that V satisfies (21) for all $\xi \in C \cup D$ where α_1 and α_2 are given in the Proof of Theorem 3.9. Note that communication noise only occurs upon communication events, when $\xi \in D$. Therefore, for each $\xi \in C$, we have that

$$\langle \nabla V(\xi), \widetilde{f}(\xi)\rangle \leq -\beta|\xi|^2_{\mathcal{A}} \leq -\frac{\beta}{\alpha_2}V(\xi) \tag{37}$$

and $\beta = -\overline{\lambda}(M(\nu))$ for each $\nu \in [0,\overline{T}]$ where M is given by (18). Moreover, at jumps, we have that the state is updated by (34), with $m_i(t) = \widetilde{\varphi}_i(x,t)$. It follows that for each $\xi \in D$ and $g \in G(\xi)$, that there exists at least one timer reseting, i.e., $\tau_i = 0$, after the jump it follows that $\tau_i^+ = \nu$ where $\nu \in [T_1^i, T_2^i]$ and $\theta_i = -Km_i$. Then, if follows that

$$V(g) \leq \exp\left(\frac{\sigma\overline{T}}{N}\right)V(\xi) + b|K|^2|m|^2 \tag{38}$$

where $b = \exp(\sigma\overline{T}/N)\overline{\lambda}(Q)$ and we use the fact that $\exp(\sigma\overline{\tau}^+) = \exp(\sigma\overline{\tau})\exp(\sigma\nu/N)$.

Now pick $\phi \in \mathcal{S}_{\mathcal{H}}$, or any $(t,j) \in \operatorname{dom}\phi$ and let $0 = t_0 \leq t_1 \leq t_2 \leq \cdots \leq t_{j+1} \leq t$ satisfy $\operatorname{dom}\phi \cup ([0,t_{j+1}] \times \{0,1,\ldots,j\}) = \cup_{s=0}^{j}([t_s,t_{s+1}] \times \{s\})$. For each $s \in \{0,1,\ldots,j\}$ and almost all $r \in [t_s,t_{s+1}]$, $\phi(r,s) \in C$. Then, integrating both sides of (37) implies that for each $s \in \{0,1,\ldots,j\}$ and for almost all $r \in [t_s,t_{s+1}]$, we have that $V(\phi(r,s)) \leq \exp\left(-\frac{\beta}{\alpha_2}\right)V(\phi(t_s,s))$. Similarly, for each $s \in \{1,2,\ldots,j\}$, $\phi(t_s,s-1) \in D$, and using (38), we get

$$V(\phi(t_s,s-1)) \leq \exp\left(\frac{\sigma\overline{T}}{N}\right)V(\phi(t_s,s-1) + b|K|^2|m|^2.$$

By using the previous two expressions, it follows that for each $(t,j) \in \operatorname{dom}\phi$, we have

$$\begin{aligned} V(\phi(t,j)) &\leq \exp\left(\frac{\sigma\overline{T}}{N}j - \frac{\beta}{\alpha_2}t\right)V(\phi(0,0)) \\ &\quad + b|m|^2_\infty \sum_{k=1}^{j}\left(\exp\left(\frac{\sigma\overline{T}}{N}k\right)\exp\left(-\frac{\beta}{\alpha_2}(t-t_k)\right)\right). \end{aligned}$$

and that the right summation of the above expression can be reduced. xNamely, first note that for $t \geq t_j$, we have $\sum_{k=1}^{j}\exp\left(\frac{\sigma\overline{T}}{N}k\right)\exp\left(-\frac{\beta}{\alpha_2}(t-t_k)\right) \leq \sum_{k=1}^{j}\exp\left(\frac{\sigma\overline{T}}{N}k\right)\exp\left(-\frac{\beta}{\alpha_2}(t_j-t_k)\right)$. Due to the increasing sequence of times $t_1 \leq t_2 \leq \cdots \leq t_j$, there must exist an integer $\tilde{j}$ which defines the maximum multiple of N, i.e., $\tilde{j} = \lfloor\frac{j}{N}\rfloor$. Then, we can group the expression $\sum_{k=1}^{j}\left(\exp\left(\frac{\sigma\overline{T}}{N}k\right)\exp\left(-\frac{\beta}{\alpha_2}(t_j-t_k)\right)\right)$ into a double sum as follows:

$$\sum_{k=1}^{j}\left(\exp\frac{\sigma\overline{T}}{N}k-\frac{\beta}{\alpha_2}(t_j-t_k)\right)$$
$$=\sum_{s=0}^{\tilde{j}-1}\sum_{k=1}^{N}\exp\left(\frac{\sigma\overline{T}}{N}(sN+k)-\frac{\beta}{\alpha_2}(t_j-t_{sN+k})\right)$$
$$+\sum_{k=\tilde{j}N+1}^{j}\exp\left(\frac{\sigma\overline{T}}{N}k-\frac{\beta}{\alpha_2}(t_j-t_k)\right)$$

Note that for each $s\in\{0,\dots,\tilde{j}-1\}$, we have

$$\max_{t_{sN+k},k\in\{1,\dots,N-1\}}\sum_{k=1}^{N}\exp\left(\frac{\sigma\overline{T}}{N}(sN+k)-\frac{\beta}{\alpha_2}(t_j-t_{sN+k})\right)$$
$$=\sum_{k=1}^{N}\exp\left(-\frac{\beta}{\alpha_2}(t_j-t_{(s+1)N})+\sigma\overline{T}(s+1)\right)$$
$$=N\exp\left(-\frac{\beta}{\alpha_2}(t_j-t_{(s+1)N})+\sigma\overline{T}(s+1)\right)$$

which corresponds to the maximizer satisfying $t_{sN+k}=t_{(s+1)N}$ for all $k\in\{1,\dots,N-1\}$. Therefore, it follows that

$$\sup_{\substack{\phi\in\mathcal{S}_\mathcal{H}\\(t_j,j)\in\operatorname{dom}\phi}}\sum_{k=1}^{j}\exp\left(\frac{\sigma\overline{T}}{N}k-\frac{\beta}{\alpha_2}(t_j-t_k)\right)$$
$$\leq\sup_{\substack{\phi\in\mathcal{S}_\mathcal{H}\\(t_j,j)\in\operatorname{dom}\phi}}\sum_{s=1}^{\tilde{j}-1}\max_{\substack{t_{sN+k}\\k\in\{1,\dots,N-1\}}}\sum_{k=1}^{N}\exp\left(\frac{\sigma\overline{T}}{N}(sN+k)\right.$$
$$\left.-\frac{\beta}{\alpha_2}(t_j-t_{sN+k})\right)$$
$$+\sup_{\substack{\phi\in\mathcal{S}_\mathcal{H}\\(t_j,j)\in\operatorname{dom}\phi}}\sum_{k=\tilde{j}N+1}^{j}\exp\left(\frac{\sigma\overline{T}}{N}k-\frac{\beta}{\alpha_2}(t_j-t_k)\right)$$
$$\leq\sup_{\substack{\phi\in\mathcal{S}_\mathcal{H}\\(t_j,j)\in\operatorname{dom}\phi}}\sum_{k=1}^{\tilde{j}-1}N\exp\left(-\frac{\beta}{\alpha_2}(t_{\tilde{j}N}-t_{(s+1)N})+\sigma\overline{T}(s+1)\right)$$
$$+N\exp(\sigma\overline{T})$$

where we use the property that $j-\tilde{j}N<N$. By item 3 in Lemma 3.3, we have that $t_{(j+1)N}-t_{jN}\in[\underline{T},\overline{T}]$ for all $j\geq 0$ such that $(t_{(j+1)N},(j+1)N),(t_{jN},jN)\in\operatorname{dom}\phi$ which implies that for each $s\in\{0,1,\dots,\tilde{j}-1\}$ $t_{\tilde{j}N}-t_{sN}\in[(\tilde{j}-s)\underline{T}),(\tilde{j}-s)\overline{T}]$. Therefore,

$$\sup_{\substack{\phi\in\mathcal{S}_\mathcal{H}\\(t_j,j)\in\operatorname{dom}\phi}}\sum_{k=1}^{j}\exp\left(\frac{\sigma\overline{T}}{N}k-\frac{\beta}{\alpha_2}(t_j-t_k)\right)$$
$$\leq N\exp(\sigma\overline{T})\sum_{s=1}^{\tilde{j}}\exp\left(\left(-\frac{\beta}{\alpha_2}\underline{T}+\sigma\overline{T}\right)s\right)+N\exp(\sigma\overline{T})$$
$$=N\exp(\sigma\overline{T})\sum_{s=0}^{\tilde{j}}\exp\left(\left(-\frac{\beta}{\alpha_2}\underline{T}+\sigma\overline{T}\right)s\right)$$

Then, it follows that

$$V(\phi(t,j))\leq\exp\left(-\frac{\beta}{\alpha_2}t+\frac{\sigma\overline{T}}{N}j\right)$$
$$+N\exp(\sigma\overline{T})b|K|^2|m|_\infty^2\sum_{s=0}^{\tilde{j}}\exp\left(\left(-\frac{\beta}{\alpha_2}\underline{T}+\sigma\overline{T}\right)s\right)$$

where $\tilde{j}=\lfloor\frac{j}{N}\rfloor$. Note that by the continuity of (19), there exists small positive scalar ϵ such that $-\frac{\beta}{\alpha_2}\underline{T}+\sigma\overline{T}\leq-\epsilon$. Note that for each $n\in\mathbb{N}$, we have that

$$\sum_{s=0}^{n}\exp(-\epsilon s)\leq\frac{\exp(-\epsilon)}{\exp(-\epsilon)-1}=:S. \tag{39}$$

Then, it follows from (21), we have (36). We can conclude the proof using similar arguments as in the proof of Theorem 3.9. ■

4 Numerical Examples

4.1 Synchronization under Nominal Conditions

Example 4.1 *Given $T_2^i=0.7$ and $T_1^i=0.9$ for each $i\in\mathcal{V}$, we apply Theorem 3.9 to a network of six harmonic oscillators, where each agent has dynamics given by*

$$\ddot{x}_i+x_i=u_i. \tag{40}$$

We consider the case where each agent is connected only to two neighbors in a circle graph. Moreover, the output of each agent is both position x_1 and velocity x_2 information, i.e, $H=I$. In state space form, we have an LTI system of the form in (1) with state matrices $A=\begin{bmatrix}0&1\\-1&0\end{bmatrix}$, $B=\begin{bmatrix}0\\1\end{bmatrix}$ and an adjacency matrix given by

$$\mathcal{G}=\begin{bmatrix}0&1&0&0&0&1\\1&0&1&0&0&0\\0&1&0&1&0&0\\0&0&1&0&1&0\\0&0&0&1&0&1\\1&0&0&0&1&0\end{bmatrix}. \tag{41}$$

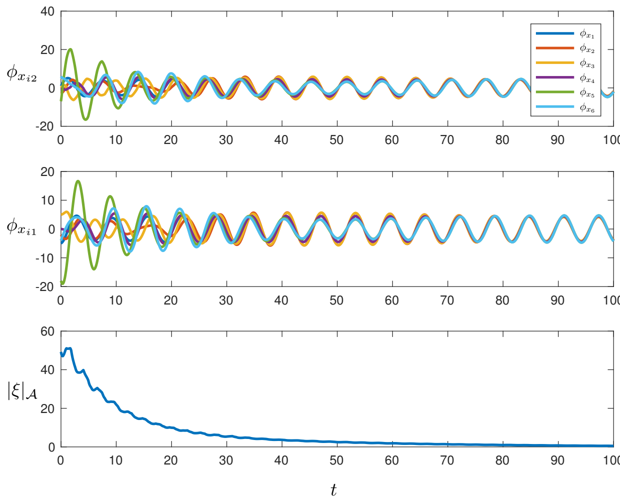

Fig. 1. Numerical solutions of 6 interconnected linear oscillators communicating over a ring graph.

It can be shown that the following parameters $K = -\begin{bmatrix}0.15 & 0.15\end{bmatrix}$, $E = -1$, $\sigma = 0.9$ *and* P_i *and* Q_i *such that*

$$P_2 = P_3 = \begin{bmatrix}0.1173 & -0.0025\\ \star & 0.1132\end{bmatrix} \qquad Q_2 = Q_3 = 0.2162$$

$$P_4 = P_5 = \begin{bmatrix}0.1144 & 0.0092\\ \star & 0.0963\end{bmatrix} \qquad Q_4 = Q_5 = 0.2097$$

$$P_6 = \begin{bmatrix}0.1116 & -0.0134\\ \star & 0.0897\end{bmatrix} \qquad Q_5 = 0.2023$$

satisfy the conditions in (18) *and* (19).[10] *In Figure 1, a numerical solution* $\phi = (\phi_x, \phi_\eta, \phi_\tau)$ *to the hybrid system* $\mathcal{H}$ *with the above parameters from initial conditions* $\phi_x(0,0) = (-5, 1, -2, -3, 5, 0, 0, 0, -18, -7, -4, 6)$, $\phi_\eta = (0.5, 0, 10, -2, 5, -10)$ *and* $\phi_\tau(0,0) = (0.25, 0.5, 0.86, 0.87, 0.14, 0.1)$ *is shown.*

The convergence to the synchronization set $\mathcal{A}$ *is exponential in nature, and is guaranteed by the sufficient condition in Proposition 3.14. In Table 1, we compare the convergence time (in flow time,* t*) of solutions to* $\mathcal{H}$ *with different gains* K *and* E *in* (9) *and* (10)*, respectively. Note the conditions in Theorem 3.9 are not necessary and it may be possible that gains can be found so that solutions still converge to the synchronization set. In Table 1, we indicate whether it is possible to satisfy the conditions in Theorem 3.9 for the gains chosen by placing a* ✓ *if the conditions are satisfied and by placing a* × *if it is not possible to satisfy the conditions for the selected gain. Moreover, in Table 1, we compare convergence times of solutions to the set* $\mathcal{A}$ *for different parameter choices. More specifically, we consider a solution* ϕ

K	E	Theorem 3.9	t^*
$-\begin{bmatrix}0.1 & 0.1\end{bmatrix}$	-0.5	✓	165.38
$-\begin{bmatrix}0.15 & 0.15\end{bmatrix}$	-1	✓	120.2
$-\begin{bmatrix}0.5 & 0.5\end{bmatrix}$	-1.8	×	30.1
$-\begin{bmatrix}0.6 & 0.6\end{bmatrix}$	-0.1	×	27.05
$\begin{bmatrix}0.15 & -0.6\end{bmatrix}$	-0.1	×	30.66

Table 1
Comparison of convergence times for different gains K and E for the hybrid system $\mathcal{H}$ with asynchronous communication in Section 3.1. The ✓ indicates that the conditions are satisfied, and the × indicates that the conditions are not satisfied but solutions converge to the synchronization set.

such that $|\phi(0,0)|_{\mathcal{A}} \approx 50$ *and find the time it takes for the solution to converge to and stay in a neighborhood near* $\mathcal{A}$ *in* (16)*, i.e., we find* t^* *such that* $t^* = \{T \in \mathbb{R}_{\geq 0} : |\phi(t,j)|_{\mathcal{A}} \leq 0.1 \ \forall (t,j) \in \operatorname{dom}\phi \text{ s.t. } t \geq T\}$*. Due to the nonuniqueness of solutions* $\mathcal{H}$ *in* (14) *and* (15) *when the network parameters are such that* $T_1^i \neq T_2^i$*, Table 1 provides an average* t^* *over 100 solutions.* △

A small-world network is a type of sparse network known to model real-world settings such as the world wide web, electric power girds, and networks of brain neurons. In particular, a small-world network is a graph structure in which most agents are, on average, a short geodesic distance[11] from any other node. In the following example, we use the random graph generator in [20] to generate the interconnection between 100 agents.

Example 4.2 *In this example, we consider the case of a network of 100 agents with dynamics as in* (40) *with* $T_1^i = 0.7$ *and* $T_2^i = 0.9$ *for each* $i \in \mathcal{V}$*. We generated a random graph using the small world generator in [20] for* $N = 100$*, the average degree* $k = 3$ *and special restructuring parameter* $\beta = 0.1$*. The resulting graph structure is depicted in the upper left of Figure 2. Furthermore, we use the parameters in Example 4.1, namely,* $K = -[0.15, 0.15]$ *and* $E = -1$*. The solutions* $\phi = (\phi_x, \phi_\eta, \phi_\tau)$ *were initialized randomly inside a bounded region, namely,* $\phi_{x_i}(0,0) \in [-5,5]^2$, $\phi_{\eta_i}(0,0) \in [-5,5]$ *and* $\phi_{\tau_i}(0,0) \in \mathcal{T}$ *for each* $i \in \mathcal{V}$[12] *The plots in the upper right section of Figure 2 show the evolution of the first component of the plant state* x_{1i} *for each* $i \in \mathcal{V}$*. It can be seen that solutions asymptotically converge to synchronization as time progresses: in fact, the bottom plot shows that, indeed, the error converges to zero.* △

[10] Code at `github.com/HybridSystemsLab/LTIAsyncSync`

[11] A geodesic distance is defined by the minimum number of edges traversed to get from the starting node to the end node.

[12] Code at `github.com/HybridSystemsLab/LTISyncSmallWorld`

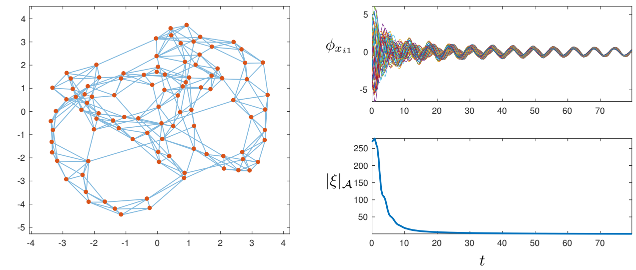

Fig. 2. (left) Randomly generated undirected small-world network containing 100 agents. (upper right) The first component of the states of each agent in the network. Note that over time all agents converge to synchrony. (bottom right) The norm of the relative error over ordinary time converges to zero.

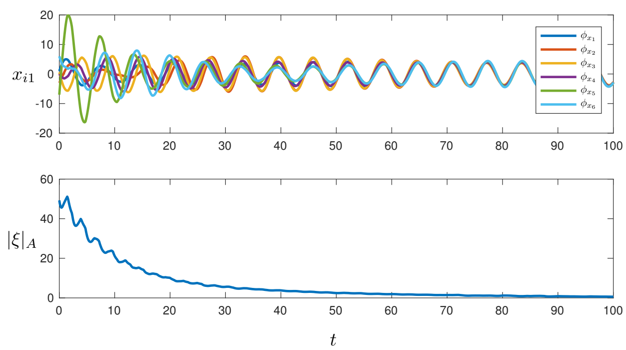

Fig. 3. When communication noise is present, solutions converge to a neighborhood about the synchronization set as indicated by the norm of the relative error converging to an average value of 0.1147.

4.2 Synchronization under Perturbations and Information Loss

Example 4.3 *In this example, we consider the case of $\mathcal{H}$ with measurement noise. Let the system be given by the dynamics in Example 4.1 connected by a network with adjacency matrix in* (41) *where $T_1^i = 0.7$ and $T_2^i = 0.9$ for each $i \in \mathcal{V}$. Let the output y_i in* (2) *be given by a constant bias, i.e., $\varphi_i(x_i, t) \equiv m_i$ for each $i \in \mathcal{V}$, where $m_1 = (0.1, 0.1)$, $m_2 = (-0.1, -0.1)$, $m_3 = (0, 0)$, $m_4 = (0.2, 0.2)$, $m_5 = (-0.15, -0.15)$, and $m_6 = (0.3, 0.3)$. Moreover, let $K = -\begin{bmatrix} 0.15 & 0.15 \end{bmatrix}$ and $E = -1$, which, as was shown in Example 4.1, satisfy Theorem 3.9 therefore the resulting hybrid system $\mathcal{H}$ with data given by* (14) *and* (15) *has $\mathcal{A}$ exponentially stable. In Figure 3, we show a numerical solution to the hybrid system from the initial conditions in Example 4.1. In this figure, it can be seen that solutions converge to a neighborhood around the synchronization set $\mathcal{A}$. Namely, it can be seen that after the transient period, the norm of the relative error $|\varepsilon|$ of the solution converges to an average value of* 0.1147 *for this case.* $\triangle$

Example 4.4 *At each communication event, a packet containing a measurement y_k is received by agent i. In*

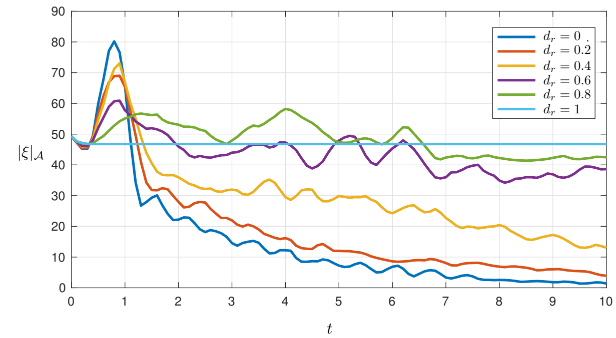

Fig. 4. The norm of the average relative error ε_i for 10 averaged trajectories for each dropout rate $d_r = \{0, 0.2, 0.4, 0.6, 0.8, 1\}$

this example, we study the robustness of the exponential stability of the set $\mathcal{A}$ in (16) *to the loss of such information, i.e., the situation when some data packets are lost. We assume that the packet arrival is given by Bernoulli random variables. Namely, a Bernoulli random variable b_k indicates whether the packet is successfully received. If it is received successfully, then $b_k = 1$; otherwise $b_k = 0$. For each $k \in \mathcal{V}$, b_k is identically and independently distributed with $P(b_k = 1) = d_r$ and $P(b_k = 0) = 1 - d_r$, where $d_r \in (0, 1)$.*

Consider the system in Example 4.1 with a graph as in (41)*. Under the same initial conditions as in Example 4.1, Figure 4 shows the norm of the average relative error $|\varepsilon|$ over the average of* 10 *trajectories projected onto the t domain for each dropout rate $d_r \in \{0, 0.2, 0.4, 0.6, 0.8, 1\}$. Note that for larger dropout rates, the convergence degrades. Moreover, for dropout rate larger than $d_r = 0.6$, the norm of the relative error ε does not appear to converge to zero.* $\triangle$

5 Conclusion

The problem of synchronization of multiple continuous-time linear time-invariant systems connected over an intermittent network was studied. Communications across the network occurs at isolated time events, which, using the hybrid systems framework was modeled using a decreasing timer. Recasting synchronization as a set stability problem, we took advantage of several properties of the graph structure and employed a Lyapunov based approach to certify exponential stability of the synchronization set. Moreover, in [24], we consider the cases when communication was synchronously occurring across the network, and, more interestingly, the case when communication was asynchronously occurring for each agent. Then, in part, as a consequence of the regularity of the hybrid systems data and the aforementioned stability properties, robustness to communication noise, and unmodeled dynamics was characterized in terms of semi-global practical stability. When communication noise was affecting the dynamics, the Lyapunov function candidate chosen certified input-to-state stability for the synchronization set and relative to such noise.

References

[1] V. N. Murthy and E. E. Fetz. Synchronization of neurons during local field potential oscillations in sensorimotor cortex of awake monkeys. *Journal of Neurophysiology*, 76(6):3968–3982, 1996.

[2] S. Phillips and R. G. Sanfelice. A framework for modeling and analysis of robust stability for spiking neurons. In *Proc. of the American Control Conf.*, pages 1414–1419, June 2014.

[3] J. A. Fax and R. M. Murray. Information flow and cooperative control of vehicle formations. *IEEE Transactions on Automatic Control*, 49(9):1465–1476, 2004.

[4] R. Olfati-Saber and R. M. Murray. Graph rigidity and distributed formation stabilization of multi-vehicle systems. In *Decision and Control, 2002, Proceedings of the 41st IEEE Conference on*, volume 3, pages 2965–2971. IEEE, 2002.

[5] R. Olfati-Saber and J. S. Shamma. Consensus filters for sensor networks and distributed sensor fusion. In *Proceedings of the Conference on Decision and Control and European Control Conference*, pages 6698–6703. IEEE, 2005.

[6] A. Sarlette, R. Sepulchre, and N. Leonard. Cooperative attitude synchronization in satellite swarms: a consensus approach. In *Proceedings of the 17th IFAC Symposium on Automatic Control in Aerospace*, pages 223–228, 2007.

[7] I. Belykh, V. Belykh, and M. Hasler. Generalized connection graph method for synchronization in asymmetrical networks. *Physica D: Nonlinear Phenomena*, 224(1–2):42 – 51, 2006.

[8] Q. Hui, W. M. Haddad, and S. P. Bhat. Finite-time semistability theory with applications to consensus protocols in dynamical networks. In *Proceedings of the American Control Conference*, pages 2411–2416, July 2007.

[9] J. J. Slotine, W. Wang, and K. El-Rifai. Contraction analysis of synchronization in networks of nonlinearly coupled oscillators. In *Proceedings of the 16th International Symposium on Mathematical Theory of Networks and Systems*, 2004.

[10] D. Angeli. A Lyapunov approach to incremental stability properties. *IEEE Transactions on Automatic Control*, 47(3):410–421, 2002.

[11] S. Cai, P. Zhou, and Z. Liu. Synchronization analysis of hybrid-coupled delayed dynamical networks with impulsive effects: A unified synchronization criterion. *Journal of the Franklin Institute*, 352(5):2065 – 2089, 2015.

[12] L. Scardovi and R. Sepulchre. Synchronization in networks of identical linear systems. *Automatica*, 45(11):2557 – 2562, 2009.

[13] L. Moreau. Stability of continuous-time distributed consensus algorithms. In *2004 43rd IEEE Conference on Decision and Control (CDC)*, volume 4, pages 3998–4003, Dec 2004.

[14] R. Olfati-Saber, J. A. Fax, and R. M. Murray. Consensus and cooperation in networked multi-agent systems. *Proceedings of the IEEE*, 95(1):215–233, Jan 2007.

[15] J. Lu, D. W. C. Ho, and J. Cao. A unified synchronization criterion for impulsive dynamical networks. *Automatica*, 46(7):1215 – 1221, 2010.

[16] T. Liu, M. Cao, C. De Persis, and J. M. Hendrickx. Distributed event-triggered control for synchronization of dynamical networks with estimators. *IFAC Proceedings Volumes*, 46(27):116 – 121, 2013.

[17] O. Demir and J. Lunze. Event-based synchronisation of multi-agent systems. *IFAC Proceedings Volumes*, 45(9):1 – 6, 2012.

[18] C. De Persis and P. Frasca. Self-triggered coordination with ternary controllers. *IFAC Proceedings Volumes*, 45(26):43 – 48, 2012.

[19] J.J.B. Biemond, N. van de Wouw, W.P.M.H. Heemels, R. G. Sanfelice, and H. Nijmeijer. Tracking control of mechanical systems with a unilateral position constraint inducing dissipative impacts. In *Proceedings of the IEEE Conference on Decision and Control*, pages 4223–4228, 2012.

[20] D. J. Watts and S. H. Strogatz. Collective dynamics of 'small-world' networks. *Nature*, 393(6684):440–442, 1998.

[21] S. Phillips and R. G. Sanfelice. Robust synchronization of interconnected linear systems over intermittent communication networks. In *Proceedings of the American Control Conference*, pages 5575–5580, 2016.

[22] S. Phillips and R.G. Sanfelice. Synchronization of two linear systems over intermittent communication networks with robustness. In *Proceedings of the IEEE Conference on Decision and Control*, pages 5569–5574, December 2015.

[23] S. Phillips, Y. Li, and R. G. Sanfelice. On distributed intermittent consensus for first-order systems with robustness. In *Proceedings of 10th IFAC Symposium on Nonlinear Control Systems*, pages 146–151, 2016.

[24] S. Phillips and R.G. Sanfelice. Robust distributed synchronization of networked linear systems with intermittent information. Technical report, University of California, Santa Cruz, 2017. https://hybrid.soe.ucsc.edu/files/preprints/TR-HSL-15-2017.pdf (pass:auto2017).

[25] C. Godsil and G.F. Royle. *Algebraic Graph Theory*. Springer New York, 2001.

[26] T. Liu, D. J. Hill, and B. Liu. Synchronization of dynamical networks with distributed event-based communication. In *2012 IEEE 51st IEEE Conference on Decision and Control*, pages 7199–7204, Dec 2012.

[27] R. Goebel, R. G. Sanfelice, and A. R. Teel. *Hybrid Dynamical Systems: Modeling, Stability, and Robustness*. Princeton University Press, New Jersey, 2012.

[28] X. Jin and J. H. Park. Adaptive sliding-mode insensitive control of a class of non-ideal complex networked systems. *Information Sciences*, 274:273 – 285, 2014.

[29] F. Ferrante, F. Gouaisbaut, R. G. Sanfelice, and S. Tarbouriech. A hybrid observer with a continuous intersample injection in the presence of sporadic measurements. In *Proceedings of the IEEE Conference on Decision and Control*, pages 5654–5659, December 2015.

[30] R. G. Sanfelice and L. Praly. Convergence of nonlinear observers on $\mathbb{R}^n$ with a Riemannian metric (Part I). *IEEE Trans. on Automatic Control*, 57(7):1709–1722, July 2012.

[31] Y. Li, S. Phillips, and R. G. Sanfelice. On distributed observers for linear time-invariant systems under intermittent information constraints. In *Proc. of 10th IFAC Symposium on Nonlinear Control Systems*, pages 666–671, 2016.

[32] Y. Li, S. Phillips, and R. G. Sanfelice. Robust distributed estimation for linear systems under intermittent information. *To appear in IEEE Transactions on Automatic Control*, 2018.

[33] S. E. Tuna. Synchronizing linear systems via partial-state coupling. *Automatica*, 44(8):2179 – 2184, 2008.

[34] A. R. Teel, F. Forni, and L. Zaccarian. Lyapunov-based sufficient conditions for exponential stability in hybrid systems. *IEEE Transactions on Automatic Control*, 58(6):1591–1596, June 2013.