Results on Incremental Stability for a Class of Hybrid Systems

Yuchun Li, Sean Phillips, and Ricardo G. Sanfelice

***Abstract*— *Incremental stability* is the notion that the distance between every pair of solutions to the system has stable behavior and approaches zero asymptotically. This paper introduces this notion for a class of hybrid systems. In particular, we define incremental stability as well as incremental partial stability, and study their properties. The approach used to derive our results consists of recasting the incremental stability problem as a set stabilization problem, for which the tools for asymptotic stability of hybrid systems are applicable. In particular, we propose an auxiliary hybrid system to study the stability of the diagonal set, which relates to incremental stability of the original system. The proposed notions are illustrated in examples throughout the paper.**

I. Introduction

Incremental stability is the notion that the distance between any two solutions to the system has stable behavior and converges to zero; see a definition for continuous-time systems in, e.g., [**?**], [1], [**?**]. Recently, incremental stability has received increased attention due to its application to synchronization [**?**], [2], [**?**], the design of observers [**?**], the design of tracking controllers [**?**], and other nonlinear control problems [2], [**?**]. Of particular interest is the definition of incremental stability in [1], where, after defining it for continuous-time systems, the author provides necessary and sufficient conditions in terms of Lyapunov functions and links incremental stability to input-to-state stability. Furthermore, due to often being miss-interpreted as a property of convergent systems, recently, the authors in [3] provide a rigorous comparison between these two properties, suggesting that neither implies the other.

Hybrid systems are dynamical systems that exhibit both continuous and discrete behavior. These types of systems are capable of modeling a wide range of dynamical systems, which include robotic, automotive, and power systems as well as natural processes. This paper considers the hybrid systems framework presented in [4], where the continuous dynamics (or flows) of a hybrid system are modeled using differential equations/inclusions, while the discrete dynamics (or jumps) are captured by difference equations/inclusions. A set stability theory in terms of Lyapunov functions is available; see [4], [5]. Motivated by the applications of incremental stability and these recent advancements in hybrid systems theory, in this paper, we introduce concepts for the study of incremental stability of hybrid systems in the framework of [4]. In particular, we have the following contributions:

1) For a class of such systems, we present notions of incremental stability and establish sufficient and necessary conditions for such a property;
2) we establish an equivalence relationship between a system being incrementally uniformly globally asymptotically stable and a set being uniformly globally asymptotically stable (for an auxiliary hybrid system);
3) Furthermore, a notion of incremental partial stability and an equivalence to uniform asymptotic stability are also proposed.

It is worth noting that this article contains new incremental stability notions for the hybrid framework in [4]-[6]. In fact, we are not aware of any previous result on incremental stability for hybrid systems. To the best of our knowledge, the notion of incremental stability and its properties for hybrid systems have not been thoroughly studied before, only discussed briefly in [**?**] for a class of transition systems in the context of bisimulations.

The remainder of this paper is organized as follows. Section II introduces the notation and some preliminaries on hybrid systems. In Section III, the notion of incremental stability is introduced for a class of hybrid systems. Necessary and sufficient conditions for incremental stability using a direct approach are also given in the same section. In Section IV, we build an auxiliary hybrid system to study the incremental stability problem. In Section V, the notion of partial incremental stability is introduced and illustrated.

II. Preliminaries

A. Notation

Given a set $S \subset \mathbb{R}^n$, the closure of S is the intersection of all closed sets containing S, denoted by $\overline{S}$. $\mathbb{R}_{\geq 0} := [0, \infty)$ and $\mathbb{N} := \{0, 1, \dots\}$. Given vectors $\nu \in \mathbb{R}^n$, $w \in \mathbb{R}^m$, $|\nu|$ defines the Euclidean vector norm $|\nu| = \sqrt{\nu^\top \nu}$, and $[\nu^\top \; w^\top]^\top$ is equivalent to (ν, w). Given a function $f : \mathbb{R}^m \to \mathbb{R}^n$, its domain of definition is denoted by dom f, i.e., dom $f := \{x \in \mathbb{R}^m : \; f(x) \text{ is defined}\}$. The range of f is denoted by rge f, i.e., rge $f := \{f(x) : x \in \operatorname{dom} f\}$. The right limit of the function f is defined as $f^+(x) := \lim_{\nu \to 0^+} f(x + \nu)$ if it exists. Given a point $y \in \mathbb{R}^n$ and a closed set $\mathcal{A} \subset \mathbb{R}^n$, $|y|_{\mathcal{A}} := \inf_{x \in \mathcal{A}} |x - y|$. A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a class-$\mathcal{K}_\infty$ function, also written $\alpha \in \mathcal{K}_\infty$, if α is zero at zero, continuous, strictly increasing, and unbounded; α is positive definite, also written $\alpha \in \mathcal{PD}$, if $\alpha(s) > 0$ for all $s > 0$ and $\alpha(0) = 0$. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a class-$\mathcal{KL}$ function, also

Y. Li, S. Phillips, and R. G. Sanfelice are with the Department of Computer Engineering, University of California, Santa Cruz, CA 95064. Email: `yli31,seaphill,ricardo@ucsc.edu` This research has been partially supported by the National Science Foundation under CAREER Grant no. ECS-1150306 and by the Air Force Office of Scientific Research under Grant no. FA9550-12-1-0366.

written $\beta \in \mathcal{KL}$, if it is nondecreasing in its first argument, nonincreasing in its second argument, $\lim_{r\to 0^+} \beta(r,s) = 0$ for each $s \in \mathbb{R}_{\geq 0}$, and $\lim_{s\to\infty} \beta(r,s) = 0$ for each $r \in \mathbb{R}_{\geq 0}$. Given a function $f : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^{m+n}$, $\nabla_x f(x,y) := \frac{\partial f}{\partial x}(x,y)$. Given a matrix $A \in \mathbb{R}^{n\times n}$, $\text{eig}(A)$ is the set of eigenvalues of A; $\lambda_{\max}(A) = \max\{\text{Re}(\lambda) : \lambda \in \text{eig}(A)\}$; $\lambda_{\min}(A) = \min\{\text{Re}(\lambda) : \lambda \in \text{eig}(A)\}$; $|A| := \max\{|\lambda|^{\frac{1}{2}} : \lambda \in \text{eig}(A^\top A)\}$.

B. Preliminaries on hybrid systems

In this paper, a hybrid system $\mathcal{H}$ has data (C, f, D, g) and is defined by

$$\begin{aligned} \dot{z} &= f(z) \quad z \in C, \\ z^+ &= g(z) \quad z \in D, \end{aligned} \tag{1}$$

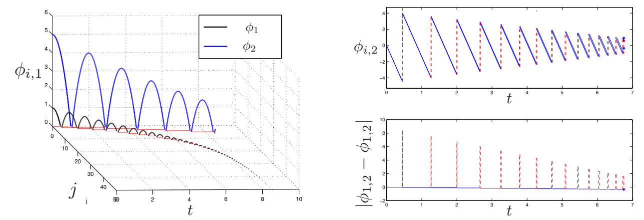
where $z \in \mathbb{R}^n$ is the state, f defines the flow map capturing the continuous dynamics and C defines the flow set on which f is effective. The map g defines the jump map and models the discrete behavior, while D defines the jump set, which is the set of points from where jumps are allowed. A solution ϕ to $\mathcal{H}$ is parametrized by $(t,j) \in \mathbb{R}_{\geq 0} \times \mathbb{N}$, where t denotes ordinary time and j denotes jump time. The domain $\text{dom}\,\phi \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ is a hybrid time domain if for every $(T,J) \in \text{dom}\,\phi$, the set $\text{dom}\,\phi \cap ([0,T] \times \{0,1,\ldots,J\})$ can be written as the union of sets $\cup_{j=0}^{J}(I_j \times \{j\})$, where $I_j := [t_j, t_{j+1}]$ for a time sequence $0 = t_0 \leq t_1 \leq t_2 \leq \cdots \leq t_J$. The t_j's with $j > 0$ define the time instants when the state of the hybrid system jumps and j counts the number of jumps. A solution to $\mathcal{H}$ is called maximal if it cannot be extended, i.e., it is not a truncated version of another solution. It is called complete if its domain is unbounded. A solution is Zeno if it is complete and its domain is bounded in the t direction. A solution is precompact if it is complete and bounded. The set $\mathcal{S}_{\mathcal{H}}$ contains all maximal solutions to $\mathcal{H}$, and the set $\mathcal{S}_{\mathcal{H}}(\xi)$ contains all maximal solutions to $\mathcal{H}$ from ξ. We consider the definition of uniform global asymptotic stability (UGAS) for a set given in [4, Definition 3.6]. A sufficient condition for a set $\mathcal{A}$ to be UGAS for $\mathcal{H}$ is given in [4, Proposition 3.27]. A hybrid system $\mathcal{H} = (C, f, D, g)$ is said to satisfy hybrid basic conditions if

(A1) C and D are closed;
(A2) $f : \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^n$ are continuous.

We refer the reader to [4] and [5] for more details on these assumptions and the hybrid systems framework.

III. Incremental stability for hybrid systems and basic properties

In this paper, for hybrid systems $\mathcal{H}$ as in (1), we are interested in characterizing the incremental stability property, namely, the notion that the distance between every pair of maximal solutions to the system has stable behavior and approaches zero asymptotically. To highlight the intricacies of this property in the hybrid setting, consider the so-called bouncing ball system. This is a canonical example of hybrid systems to which every maximal solution is Zeno and converges to the origin; see [4, Example 1.1 and 2.12] for more details. Consider two solutions to this system, given



(a) The first component (height) of solutions from $\phi_1(0,0) = (0,1)$ and $\phi_2(0,0) = (5,0)$.

(b) The second component (velocity) of solutions from $\phi_1(0,0) = (1,0)$ and $\phi_2(0,0) = (1.01, 0.01)$.

Fig. 1. Two solutions ϕ_1 and ϕ_2 to the bouncing ball system. Note that while both solutions converge to zero, the pointwise Euclidean distance between them does not decrease monotonically.

by ϕ_1 and ϕ_2, from initial conditions $\phi_1(0,0) = (1,0)$ (ball initialized close to the ground with zero initial velocity) and $\phi_2(0,0) = (5,0)$ (ball initialized at a larger height). Figure 1(a) shows the position component $(\phi_{i,1})$ of these two solutions. The Zeno behavior of the solutions makes it extremely difficult to analyze incremental stability. In fact, since incremental stability requires comparing the distance between two solutions, if one solution (ϕ_1) reaches the Zeno time sooner than the second solution (ϕ_2), then the Euclidean distance between these two solutions cannot be evaluated forward after the solution has reached Zeno. This situation is shown in Figure 1(a), where ϕ_1 approaches Zeno at about $t = 8.5$ while ϕ_2 is still describing the motion of the ball bouncing. Even if the solutions start from nearby initial conditions, the point-wise distance between them is not monotonically decreasing before the Zeno time; e.g. see Figure 1(b). In a more extreme situation, if solutions ϕ_1 and ϕ_2 are initialized at $\phi_1(0,0) = (0,0)$ and $\phi_2(0,0) = (1,0)$, respectively, then ϕ_1 is a solution that only jumps and never evolves in the t direction, while ϕ_2 evolves in both directions (t and j) until it reaches a Zeno solution. This extreme difference in the domains of the solutions makes it difficult (if not impossible) to compare ϕ_1 and ϕ_2.

To avoid such issues, we consider the class of hybrid systems satisfying the following assumption.

Assumption 3.1: The hybrid system $\mathcal{H} = (C, f, D, g)$ is such that

1) each maximal solution to $\mathcal{H}$ has a hybrid time domain that is unbounded in the t direction;
2) each solution to $\mathcal{H}$ does not have two consecutive jumps without flow in between.

The following result from [6] provides a sufficient condition for condition 2).

Lemma 3.2: [6, Lemma 2.7] Suppose that $\mathcal{H}$ satisfies the hybrid basic conditions and $g(D) \cap D = \emptyset$. Then, for any precompact solution $\phi \in \mathcal{S}_{\mathcal{H}}$ there exists $\gamma > 0$ such that $t_{j+1} - t_j \geq \gamma$ for all $j \geq 1$, $(t_j, j), (t_{j+1}, j) \in \text{dom}\,\phi$.

Assumption 3.1 can be checked using Lemma 3.2 and [4, Proposition 6.10].

Now, we define the notion of incremental stability for a hybrid system.

Definition 3.3: Consider a hybrid system $\mathcal{H}$ and suppose Assumption 3.1 holds. The hybrid system $\mathcal{H}$ is said to be

- *incrementally uniformly globally stable* (δUGS) if there exists a class-$\mathcal{K}_\infty$ function α such that any two maximal solutions ϕ_1, ϕ_2 to $\mathcal{H}$ satisfy
$$|\phi_1(t,j_1) - \phi_2(t,j_2)| \le \alpha(|\phi_1(0,0) - \phi_2(0,0)|) \quad (2)$$
for all $(t,j_1) \in \text{dom } \phi_1$ and $(t,j_2) \in \text{dom } \phi_2$;
- *incrementally uniformly globally attractive* (δUGA) if for each $\varepsilon > 0$ and $r > 0$ there exists $T > 0$ such that for any two maximal solutions ϕ_1 and ϕ_2 to $\mathcal{H}$ with
$$|\phi_1(0,0) - \phi_2(0,0)| \le r,$$
$(t,j_i) \in \text{dom } \phi_i$ for each $i \in \{1,2\}$, $t \ge T$ implies that
$$|\phi_1(t,j_1) - \phi_2(t,j_2)| \le \varepsilon;$$
- *incrementally uniformly globally asymptotically stable* (δUGAS) if it is both incrementally uniformly globally stable and incrementally uniformly globally attractive.

Nonuniform versions of these notions are also possible (see Section V for such notions for the partial stability case).

The following necessary condition for δUGAS is immediate.

Lemma 3.4: Suppose $\mathcal{H}$ satisfies Assumption 3.1 and is δUGAS. Then, every maximal solution to $\mathcal{H}$ is unique.

The following result establishes a $\mathcal{KL}$ characterization for δUGAS. This notion is similar to that introduced in [1, Definition 2.1].

Lemma 3.5: Consider a hybrid system $\mathcal{H}$ satisfying Assumption 3.1. Then, $\mathcal{H}$ is δUGAS if and only if there exists a function $\beta \in \mathcal{KL}$ such that for any two maximal solutions ϕ_1 and ϕ_2 to $\mathcal{H}$ from $\phi_1(0,0) = \xi$ and $\phi_2(0,0) = \eta$, respectively,
$$|\phi_1(t,j_1) - \phi_2(t,j_2)| \le \beta(|\xi - \eta|, t), \quad (3)$$
for all $(t,j_i) \in$ dom ϕ_i, for each $i \in \{1,2\}$.

Note that the solutions to a hybrid system $\mathcal{H}$ are parameterized by both t and j, but the $\mathcal{KL}$ bound in Lemma 3.5 only depends on t. While the bound might be conservative since it does not involve j, at this time it is not obvious how to make the bound tighter. To illustrate this point, consider the hybrid system with the following data:
$$\begin{aligned} f(x) &:= -x \qquad \forall\, x \in C := [-1,0] \\ g(x) &:= \alpha x \qquad \forall\, x \in D := [0,1] \end{aligned} \quad (4)$$
with $\alpha \in (0,1)$. Consider two maximal solutions ϕ_1 and ϕ_2 with initial conditions $\phi_1(0,0) = -0.5$ and $\phi_2(0,0) = 0.5$, respectively. Each solution can be computed analytically and, for $\alpha = 0.5$, they are given by $\phi_1(t,0) = \phi(0,0)\exp(-t)$ for each $t \in \mathbb{R}_{\ge 0}$ and $\phi_2(0,j) = \phi_2(0,0)(0.5)^j$ for each $j \in \mathbb{N}$. Note that both solutions approach the origin. Figure 2 depicts these two solutions. Due to the dramatically different domains of ϕ_1 and ϕ_2, it is very difficult (if not impossible) to measure their distance using a $\mathcal{KL}$ bound that depends on t and j. This is one of the reasons we study incremental stability of hybrid systems using a notion that prioritizes the flow time t.

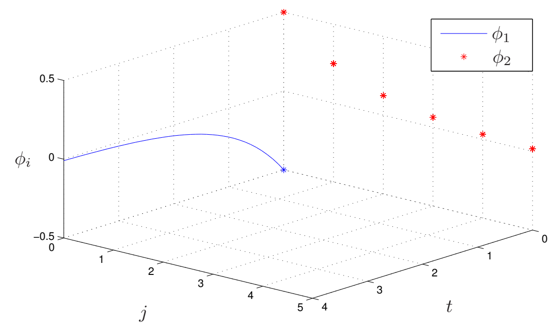


Fig. 2. Solutions ϕ_1 and ϕ_2 to illustrate the hybrid time domain. Note that both solutions converge to zero.

The following example illustrates the notion in Definition 3.3.

Example 3.6: Consider a hybrid system $\mathcal{H}$ with data
$$f(z) := -z \qquad \forall\, z \in C := \bigcup_{i \in \{0,2,4,\dots\}} [i, i+1] \quad (5)$$
$$g(z) := z - 1 \qquad \forall\, z \in D := \{2,4,\dots\}. \quad (6)$$
Note that every maximal solution to $\mathcal{H}$ is complete, bounded and does not have two consecutive jumps without flow time in between since, in particular, $D \cap g(D) = \emptyset$. Given $\varepsilon, r > 0$, consider two maximal solutions ϕ_1 and ϕ_2 from $\phi_1(0,0) = \xi$ and $\phi_2(0,0) = \eta$ such that $|\xi - \eta| < r$, it follows that these solutions are bounded by
$$|\phi_i(t,j_i)| \le e^{-t}|\phi_i(0,0)|,$$
for all $(t,j_i) \in$ dom ϕ_i and $i \in \{1,2\}$. Then, there exist $T > 0$ such that $|\phi_i(t,j_i)| \le \frac{1}{2}\varepsilon$ for each $i \in \{1,2\}$ and each $t \ge T$ with $(t,j_i) \in$ dom ϕ_i. It follows that $|\phi_1(t,j_1) - \phi_2(t,j_2)| \le \varepsilon$ for each $t \ge T$ with $(t,j_i) \in$ dom ϕ_i with $i \in \{1,2\}$, which implies that this system is δUGA. $\triangle$

Motivated by Example 3.6, the next result establishes a stability property of a set for $\mathcal{H}$ that is implied by δUGAS. For a hybrid system $\mathcal{H}$, a set $\mathcal{M} \subset \mathbb{R}^n$ is weakly forward invariant if for every $\xi \in \mathcal{M}$, there exists at least one complete $\phi \in \mathcal{S}_{\mathcal{H}}(\xi)$ with rge $\phi \subset \mathcal{M}$; see, e.g., [4, Definition 6.19]. The set $\mathcal{M}$ is strongly forward pre-invariant if for every $\phi \in \mathcal{S}_{\mathcal{H}}(\mathcal{M})$, rge $\phi \subset \mathcal{M}$; see, e.g., [4, Definition 6.25].

Theorem 3.7: Suppose $\mathcal{H}$ satisfies Assumption 3.1 and is δUGAS. Then, a compact set $\mathcal{A}$ that contains a nonempty weakly forward invariant set for $\mathcal{H}$ is uniformly globally attractive for $\mathcal{H}$. Furthermore, if $\mathcal{H}$ satisfies the hybrid basic conditions and the compact set $\mathcal{A}$ is strongly forward invariant then $\mathcal{A}$ is UGAS for $\mathcal{H}$.

The UGAS property guaranteed by Theorem 3.7 also holds when, instead of $\mathcal{A}$ being strongly forward invariant, $\mathcal{A}$ is a singleton. Furthermore, when a singleton set $\mathcal{A}$ is uniformly globally attractive for $\mathcal{H}$, the following result can be established. In particular, it justifies the property illustrated in Example 3.6.

Lemma 3.8: Consider a hybrid system $\mathcal{H}$ satisfying Assumption 3.1. Suppose that a singleton set $\mathcal{A}$ is uniformly globally attractive for $\mathcal{H}$. Then, the system $\mathcal{H}$ is incrementally uniformly globally attractive (δUGA).

IV. Incremental stability via an auxiliary hybrid system

Following the approach in [1] for continuous-time systems, we recast incremental stability as the problem of uniformly globally asymptotically stabilizing an appropriately defined hybrid system to the diagonal set. Towards that end, consider the set

$$\widetilde{\mathcal{A}} := \{\tilde{z} \in \mathbb{R}^{2n} : \tilde{z} = (z, z),\ z \in \mathbb{R}^n\}$$

for the auxiliary hybrid system $\widetilde{\mathcal{H}}$ with state $\tilde{z} = (\tilde{z}_1, \tilde{z}_2)$ and given by the following data:

$$\begin{aligned} \dot{\tilde{z}} &= \widetilde{f}(\tilde{z}) \quad \tilde{z} \in \widetilde{C}, \\ \tilde{z}^+ &= \widetilde{g}(\tilde{z}) \quad \tilde{z} \in \widetilde{D}, \end{aligned} \tag{7}$$

where $\widetilde{C} := C \times C$, $\widetilde{D} := (D \times (C \cup D)) \bigcup ((C \cup D) \times D)$, $\widetilde{f}(\tilde{z}) = (f(\tilde{z}_1), f(\tilde{z}_2))$, and $\widetilde{g}(\tilde{z}) = (\widetilde{g}_1(\tilde{z}_1), \widetilde{g}_2(\tilde{z}_2))$ with

$$\begin{aligned} \widetilde{g}_1(\tilde{z}_1) &= \begin{cases} g(\tilde{z}_1) & \tilde{z}_1 \in D \\ \tilde{z}_1 & \text{otherwise} \end{cases} \\ \widetilde{g}_2(\tilde{z}_2) &= \begin{cases} g(\tilde{z}_2) & \tilde{z}_2 \in D \\ \tilde{z}_2 & \text{otherwise} \end{cases} . \end{aligned} \tag{8}$$

The auxiliary system $\widetilde{\mathcal{H}}$ consists of two copies of the original hybrid system $\mathcal{H}$.

Remark 4.1: Though the original hybrid system $\mathcal{H}$ may satisfy the hybrid basic conditions in [4], the auxiliary system $\widetilde{\mathcal{H}}$ may not. If the hybrid basic conditions were to be satisfied for $\widetilde{\mathcal{H}}$, the jump map would have another entry given by $\widetilde{g}_i(\tilde{z}) = \{g(\tilde{z}_i), \tilde{z}_i\}$ when $\tilde{z}_i$ is in the boundary of the jump set D. Including this entry in the jump map would not be advantageous since it could introduce extra solutions to the auxiliary system. In fact, it may introduce a Zeno solution: consider the case where $\tilde{z}_1$ and $\tilde{z}_2$ are in the boundary of the jump set; then, both components of the solution could jump as $\widetilde{g}(\tilde{z}) = \tilde{z} \in D \times D$ and remain there forever. To avoid this issue, we do not insist that $\widetilde{\mathcal{H}}$ satisfies the hybrid basic conditions.

Lemma 4.2: Suppose $\mathcal{H}$ satisfies Assumption 3.1 and the hybrid basic conditions. Then, $\widetilde{\mathcal{H}}$ does not have Zeno solutions.

It is worth noting that, in general, the domain of solutions z_1 and z_2 of $\mathcal{H}$ with $z_1(0,0) = \xi$ and $z_2(0,0) = \eta$ are different from the domain of the associated solution $\tilde{z}$ of $\widetilde{\mathcal{H}}$ with $\tilde{z}(0,0) = (\xi, \eta)$. The following lemma characterizes the relationship between solutions to $\mathcal{H}$ and solutions to $\widetilde{\mathcal{H}}$.

Lemma 4.3: Suppose the hybrid system $\mathcal{H}$ satisfies Assumption 3.1. Let the sets $\mathcal{S}_{\mathcal{H}}$ and $\mathcal{S}_{\widetilde{\mathcal{H}}}$ contain all maximal solutions to $\mathcal{H}$ and $\widetilde{\mathcal{H}}$, respectively. Then, the following hold:

1) *Given a solution $\tilde{z} \in \mathcal{S}_{\widetilde{\mathcal{H}}}$ with $\tilde{z}(0,0) = (\xi, \eta)$, $\xi, \eta \in \mathbb{R}^n$, there exist $z_1, z_2 \in \mathcal{S}_{\mathcal{H}}$, $z_1(0,0) = \xi$ and $z_2(0,0) = \eta$ such that $\tilde{z}(t,j) = (z_1(t,j_1), z_2(t,j_2))$ for every $(t,j) \in \operatorname{dom} \tilde{z}$, where $(t,j_1) \in$ dom z_1 and $(t,j_2) \in$ dom z_2;*
2) *Given solutions $z_1, z_2 \in \mathcal{S}_{\mathcal{H}}$ with $z_1(0,0) = \xi$ and $z_2(0,0) = \eta$, $\xi, \eta \in \mathbb{R}^n$, there exists $\tilde{z} \in \mathcal{S}_{\widetilde{\mathcal{H}}}$ with $\tilde{z}(0,0) = (\xi, \eta)$ such that*

$$\tilde{z}(t,j) = (z_1(t,j_1), z_2(t,j_2))$$

for every $(t,j_1) \in$ dom z_1 and $(t,j_2) \in$ dom z_2, where $(t,j) \in$ dom $\tilde{z}$.

With the equivalence between solutions to $\mathcal{H}$ and $\widetilde{\mathcal{H}}$ established in Lemma 4.3, we are ready to present the following result linking δUGAS of $\mathcal{H}$ and UGAS of the diagonal for $\widetilde{\mathcal{H}}$.

Theorem 4.4: Consider a hybrid system $\mathcal{H}$ on $\mathbb{R}^n$ satisfying Assumption 3.1. Then, $\mathcal{H}$ is δUGAS if and only if $\widetilde{\mathcal{A}} = \{(\tilde{z}_1, \tilde{z}_2) \in \mathbb{R}^n \times \mathbb{R}^n : \tilde{z}_1 = \tilde{z}_2\}$ is UGAS for the auxiliary hybrid system $\widetilde{\mathcal{H}} = (\widetilde{C}, \widetilde{f}, \widetilde{D}, \widetilde{g})$.

To illustrate the necessity of this theorem, we refer to Example 3.6. In this example, it was found that $\mathcal{H}$ is δUGAS and that Assumption 3.1 holds. Then, from Theorem 4.4, the auxiliary hybrid system $\widetilde{\mathcal{H}}$ has $\widetilde{\mathcal{A}}$ UGAS. The sufficiency of Theorem 4.4 will be discussed in Section VI.

Following directly from Theorem 4.4, the subsequent result considers the case when a subset of $\widetilde{\mathcal{A}}$ is UGAS for $\widetilde{\mathcal{H}}$.

Corollary 4.5: Consider a hybrid system $\mathcal{H}$ satisfying Assumption 3.1. Then, $\mathcal{H}$ is δUGA if a nonempty closed set $\mathcal{A}_s \subset \widetilde{\mathcal{A}} = \{(\tilde{z}_1, \tilde{z}_2) \in \mathbb{R}^n \times \mathbb{R}^n : \tilde{z}_1 = \tilde{z}_2\}$ is UGAS for the auxiliary hybrid system $\widetilde{\mathcal{H}} = (\widetilde{C}, \widetilde{f}, \widetilde{D}, \widetilde{g})$.

In light of Corollary 4.5, tools for the study of asymptotic stability can be applied to $\widetilde{\mathcal{H}}$ to establish δUGAS of $\mathcal{H}$. In Section VI, a preliminary sufficient condition for δUGAS of $\mathcal{H}$ using such tools is discussed.

V. Incremental partial stability

Hybrid systems typically have states containing logic variables, timers, etc. The solution components corresponding to these particular variables may never converge to each other. For example, consider a hybrid system with a continuous state and a timer. The timer state, τ, would evolve according to the following dynamics: $\dot{\tau} = 1$ for all $\tau \in [0,1]$ and $\tau^+ = 0$ when $\tau = 1$. Note that, while the continuous states of a pair of solutions may exhibit incremental stability behavior, the error between the timer components of the solutions to such system will never converge to zero. To provide an incremental stability property that fits to such hybrid systems, we introduce the following incremental partial stability notion.

Definition 5.1: (Incremental partial stability) Consider a hybrid system $\mathcal{H}$ with state $z \in \mathbb{R}^n$ satisfying Assumption 3.1. Let $z = (x, \nu)$, where $x \in \mathbb{R}^r$, $\nu \in \mathbb{R}^{n-r}$ and $1 \leq r \leq n$. The hybrid system $\mathcal{H}$ is said to be

- incrementally partially stable (δPS) with respect to x if for every $\varepsilon > 0$ there exists $\delta > 0$ such that, for any two maximal solutions $\phi_1 = (\phi_1^x, \phi_1^\nu)$ and $\phi_2 = (\phi_2^x, \phi_2^\nu)$ to $\mathcal{H}$,

$$|\phi_1(0,0) - \phi_2(0,0)| \leq \delta$$

implies

$$|\phi_1^x(t, j_1) - \phi_2^x(t, j_2)| \leq \varepsilon$$

for all $(t, j_i) \in \operatorname{dom} \phi_i$, $i \in \{1, 2\}$.

- incrementally globally partially attractive (δGPA) with respect to x if any two maximal solutions $\phi_1 = (\phi_1^x, \phi_1^\nu)$, $\phi_2 = (\phi_2^x, \phi_2^\nu)$ to $\mathcal{H}$ satisfies
$$\lim_{t\to\infty, (t,j_1)\in \operatorname{dom}\phi_1, (t,j_2)\in\operatorname{dom}\phi_2} |\phi_1^x(t, j_1) - \phi_2^x(t, j_2)| = 0$$
- incrementally globally asymptotically partially stable (δGAPS) with respect to x if it is both δPS and δGPA.

The following examples exercise the δPS and δGPA notions.

Example 5.2: Consider a hybrid system with state $z = (x, q) \in [-1, 1] \times \{-1, 1\}$. Its flow and jump maps are

$$f(z) := \begin{bmatrix} q \\ 0 \end{bmatrix}, \quad g(z) := \begin{bmatrix} x \\ -q \end{bmatrix}, \tag{9}$$

while its flow and jump sets are

$$C := [-1, 1] \times \{1, 2\}$$

and

$$D := (\{-1\} \times \{-1\}) \cup (\{1\} \times \{1\}),$$

respectively. Note that, for the case of $q(0, 0) = -1$ and up until the second jump (i.e., $(t_2, 1)$), a solution from $\phi(0, 0) = (\phi^x(0, 0), -1) \in C$ is given by $\phi^x(t, 0) = -t + \phi^x(0, 0)$ and $\phi^q(t, 0) = -1$ for all $t \in [0, 1 + \phi^x(0, 0)]$, and $\phi^x(t, 1) = t - \phi^x(0, 0) - 2$, $\phi^q(t, 1) = 1$ for all $t \in [1 + \phi^x(0, 0), 3 + \phi^x(0, 0)]$. Given $\varepsilon > 0$, consider any two solutions given by $\phi_1 = (\phi_1^x, \phi_1^q)$ and $\phi_2 = (\phi_2^x, \phi_2^q)$ to $\mathcal{H}$ from $\phi_1(0, 0)$ and $\phi_2(0, 0)$, pick $\delta \in (0, \varepsilon]$ such that $|\phi_1(0, 0) - \phi_2(0, 0)| \leq \delta$. Without loss of generality, assume that $\phi_1^x(0, 0) > \phi_2^x(0, 0)$ and that δ is small enough to yield $\phi_1^q(0, 0) = \phi_2^q(0, 0) = -1$. Then

$$\begin{aligned}|\phi_1^x(t, 0) - \phi_2^x(t, 0)| &= | - t + \phi_1^x(0, 0) - (-t + \phi_2^x(0, 0))| \\ &= |\phi_1^x(0, 0) - \phi_2^x(0, 0)| \leq \delta \leq \varepsilon\end{aligned}$$

for $t \in [0, 1 + \phi_2^x(0, 0)]$. Since ϕ_2 jumps first, it follows that after the jump $\phi_2^q(t, 1) = 1$ and for $t \in [1 + \phi_2^x(0, 0), 1 + \phi_1^x(0, 0)]$ we have that

$$\begin{aligned}|\phi_1^x(t, 0) - \phi_2^x(t, 1)| &= | - t + \phi_1^x(0, 0) - (t - \phi_2^x(0, 0) - 2)| \\ &\leq | - 2(1 + \phi_1^x(0, 0)) + 2 + \phi_1^x(0, 0) + \phi_2^x(0, 0)| \\ &\leq | - \phi_1^x(0, 0) + \phi_2^x(0, 0)| \leq \delta \leq \varepsilon.\end{aligned}$$

Then, it follows that

$$\begin{aligned}|\phi_1^x(t, 1) - \phi_2^x(t, 1)| &= |t - \phi_1^x(0, 0) - 2 - (t - \phi_2^x(0, 0) - 2)| \\ &= | - \phi_1^x(0, 0) + \phi_2^x(0, 0)| \leq \delta \leq \varepsilon\end{aligned}$$

for $t \in [1 + \phi_1^x(0, 0), 3 + \phi_2^x(0, 0)]$. Therefore, the system is δPS. △

Example 5.3: Consider a hybrid system with state $z = (x, q, \tau, k) \in \mathcal{X} := \mathbb{R} \times [-M, M] \times \mathbb{R}_{\geq 0} \times \mathbb{N} \setminus \{0\}$ for $M > 0$ with the following data:

$$\begin{aligned}f(z) &= (-x + q, 0, 1, 0) \quad \forall\ z \in C := \{z \in \mathcal{X} : \tau \leq k\}, \\ g(z) &= (-x, -\gamma q, 0, k + 1) \quad \forall\ z \in D := \{z \in \mathcal{X} : \tau = k\},\end{aligned}$$

where the parameter $\gamma \in (0, 1)$. For a maximal solution ϕ to the system from $\phi(0, 0) = (\phi^x(0, 0), \phi^q(0, 0), \phi^\tau(0, 0), \phi^k(0, 0))$, the ϕ^x component of the solution before the first jump is given by $\phi^x(t, 0) = \exp(-t)\phi^x(0, 0) + (1 - \exp(-t))\phi^q(0, 0)$ for all $(t, 0) \in \operatorname{dom} \phi$. Moreover, generalizing this expression to each $(t, j), (t_j, j) \in \operatorname{dom}\phi$, we obtain $\phi^x(t, j) = (-1)^j \exp(-t + t_j)\phi^x(t_j, j) + (1 - \exp(-t + t_j))\phi^q(t_j, j)$. Due to the form of the jump map for q, it follows that $\lim_{t_j\to\infty, (t_j, j)\in \operatorname{dom}\phi} \phi^q(t_j, j) = 0$. Then, we have $\lim_{t\to\infty, (t,j)\in\operatorname{dom}\phi} \phi^x(t, j) = 0$. Therefore, the system is δGPA with respect to x.

As we will show, this system is not δPS with respect to x. To draw a contradiction, assume that it has such property. For a given $\varepsilon > 0$ pick two solutions $\phi_1 = (\phi_1^x, \phi_1^q, \phi_1^\tau, \phi_1^k)$ and $\phi_2 = (\phi_2^x, \phi_2^q, \phi_2^\tau, \phi_2^k)$ to the system from $\phi_1(0, 0) = (\phi_1^x(0, 0), \phi_1^q(0, 0), \phi_1^\tau(0, 0), \phi_1^k(0, 0))$ and $\phi_2(0, 0) = (\phi_2^x(0, 0), \phi_2^q(0, 0),$
$\phi_2^\tau(0, 0), \phi_2^k(0, 0))$ such that the stability criterion is satisfied, namely, there exists $\delta > 0$ such that $|\phi_1^x(t, j_1) - \phi_2^x(t, j_2)| < \varepsilon$ for all $(t, j_1) \in \operatorname{dom}\phi_1$ and $(t, j_2) \in \operatorname{dom}\phi_2$ when $|\phi_1(0, 0) - \phi_2(0, 0)| < \delta$. Note that we can choose δ small enough so that $\phi_1^k(0, 0) = \phi_2^k(0, 0)$. Let $t_1 = \min_{(t,1)\in\text{dom } \phi_1, (t,1)\in\text{dom } \phi_2} t$. Then, we have that for all $0 \leq t \leq t_1$ and for each $i \in \{1, 2\}$

$$\phi_i^x(t, 0) = \exp(-t)\phi_i^x(0, 0) + (1 - \exp(-t))\phi_i^q(0, 0).$$

Suppose $\phi_1^x(0, 0) = \phi_2^x(0, 0) > \frac{1}{2}\exp(t_1)\varepsilon$ and $\phi_1^q(0, 0) = \phi_2^q(0, 0) = 0$ and $\phi_1^\tau(0, 0) > \phi_2^\tau(0, 0)$ such that $|\phi_1(0, 0) - \phi_2(0, 0)| \leq \delta$. Then, after the first jump (ϕ_1 in this case), we have $|\phi_1^x(t_1, 1) - \phi_2^x(t_1, 0)| > \varepsilon$, which is a contradiction. However, note that if the jump map for x is changed to $x^+ = x$, then the system is δGAPS with respect to x. △

Following the ideas in Section IV, the incremental partial stability property can also be studied in terms of an auxiliary hybrid system $\widetilde{\mathcal{H}}$. The following result establishes a relationship between GAS of a diagonal set for $\widetilde{\mathcal{H}}$ and δGAPS for $\mathcal{H}$.

Theorem 5.4: Consider a hybrid system $\mathcal{H}$ on $\mathbb{R}^n$ with state $z = (x, \nu)$, $x \in \mathbb{R}^r$, $\nu \in \mathbb{R}^{n-r}$, and $1 \leq r \leq n$, satisfying Assumption 3.1. Then, $\mathcal{H}$ is δGAPS if the set

$$\begin{aligned}\widetilde{\mathcal{A}} := \{\tilde{z} \in \mathbb{R}^n \times \mathbb{R}^n : \tilde{z} = (\tilde{x}_1, \tilde{\nu}_1, \tilde{x}_2, \tilde{\nu}_2),\ \tilde{x}_1 = \tilde{x}_2, \\ \tilde{x}_1, \tilde{x}_2 \in \mathbb{R}^r, \nu_1, \nu_2 \in \mathbb{R}^{n-r}\}\end{aligned}$$

is GAS to the auxiliary hybrid system $\widetilde{\mathcal{H}} = (\widetilde{C}, \widetilde{f}, \widetilde{D}, \widetilde{g})$.

VI. Discussion

Section IV argues that asymptotic stability of a set $\widetilde{\mathcal{A}}$ for the auxiliary hybrid system implies incremental stability of the original system. To show asymptotic stability of the set $\widetilde{\mathcal{A}}$ for $\widetilde{\mathcal{H}}$, one can employ sufficient conditions in terms of Lyapunov functions; e.g., [4, Proposition 3.27]. In fact, consider the hybrid system $\mathcal{H} = (C, f, D, g)$ which satisfies Assumption 3.1 and $\widetilde{\mathcal{A}} = \{(\tilde{z}_1, \tilde{z}_2) \in \mathbb{R}^n \times \mathbb{R}^n : \tilde{z}_1 = \tilde{z}_2\}$. Suppose there exists a Lyapunov function candidate[1] $V : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ for $\widetilde{\mathcal{H}} = (\widetilde{C}, \widetilde{f}, \widetilde{D}, \widetilde{g})$ in (7) and there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and a continuous $\rho_c \in \mathcal{PD}$ such that

$$\alpha_1(|\tilde{z}_1 - \tilde{z}_2|) \leq V(\tilde{z}_1, \tilde{z}_2) \leq \alpha_2(|\tilde{z}_1 - \tilde{z}_2|) \quad \forall\ \tilde{z} = (\tilde{z}_1, \tilde{z}_2) \in \widetilde{C} \cup \widetilde{D} \cup \widetilde{g}(\widetilde{D}); \tag{10a}$$

$$\langle \nabla_{\tilde{z}_1} V(\tilde{z}_1, \tilde{z}_2), f(\tilde{z}_1)\rangle + \langle \nabla_{\tilde{z}_2} V(\tilde{z}_1, \tilde{z}_2), f(\tilde{z}_2)\rangle \leq - \rho_c(|\tilde{z}_1 - \tilde{z}_2|) \quad \forall\ \tilde{z} = (\tilde{z}_1, \tilde{z}_2) \in \widetilde{C}; \tag{10b}$$

$$V(g(\tilde{z}_1), \tilde{z}_2) - V(\tilde{z}_1, \tilde{z}_2) \leq 0 \quad \forall\ \tilde{z}_1 \in D, \tilde{z}_2 \in C \setminus D; \tag{10c}$$

$$V(g(\tilde{z}_1), g(\tilde{z}_2)) - V(\tilde{z}_1, \tilde{z}_2) \leq 0 \quad \forall\ \tilde{z}_1, \tilde{z}_2 \in D, \tag{10d}$$

where V is symmetric about its arguments, i.e., $V(\tilde{z}_1, \tilde{z}_2) = V(\tilde{z}_2, \tilde{z}_1)$ for all $\tilde{z}_1, \tilde{z}_2 \in \mathbb{R}^n$. If, for each $r > 0$, there exist $\gamma_r \in \mathcal{K}_\infty$, $N_r > 0$ such that for every solution ϕ to $\widetilde{\mathcal{H}}$, $|\phi(0,0)|_{\widetilde{\mathcal{A}}} \in (0, r]$, $(t, j) \in \operatorname{dom} \phi$, $t + j \geq T$ imply $t \geq \gamma_r(T) - N_r$, then $\widetilde{\mathcal{A}}$ is uniformly globally asymptotically stable for $\widetilde{\mathcal{H}}$. Furthermore, $\mathcal{H}$ is δUGAS. Note that it is not evident that such conditions in (10) are necessary, as $\widetilde{\mathcal{H}}$ does not satisfy the hybrid basic conditions.

The conditions above, while potentially useful, appear to be very restrictive. The following example illustrates this fact, and motivates future development of sufficient conditions for δUGAS.

Example 6.1: Consider a hybrid system $\mathcal{H} = (C, f, D, g)$ with a linear flow map and jump map, i.e.,

$$\begin{aligned} f(z) &= Az \qquad \forall\ z \in C \\ g(z) &= Bz \qquad \forall\ z \in D \end{aligned}$$

satisfying Assumption 3.1 and having solutions that, for each $r > 0$, there exist $\gamma_r \in \mathcal{K}_\infty$, $N_r > 0$ such that for every solution ϕ to $\widetilde{\mathcal{H}}$, $|\phi(0,0)|_{\widetilde{\mathcal{A}}} \in (0, r]$, $(t, j) \in \operatorname{dom} \phi$, $t + j \geq T$ imply $t \geq \gamma_r(T) - N_r$. Following the construction of the auxiliary system $\widetilde{\mathcal{H}}$ in (7), we want to determine the conditions under which the set $\widetilde{\mathcal{A}}$ is UGAS for $\widetilde{\mathcal{H}}$. Consider the Lyapunov function candidate $V(\tilde{z}) = (\tilde{z}_1 - \tilde{z}_2)^\top P(\tilde{z}_1 - \tilde{z}_2)$, where P is a positive definite symmetric matrix that satisfies

$$\begin{aligned} PA + A^\top P &< 0, \\ B^\top PB - P &\leq 0, \end{aligned}$$

and, for each $\eta_1 \in D$ and $\eta_2 \in C \setminus D$,

$$-\eta_1^\top (B^\top P - P)\eta_2 \leq 0.$$

Then, the function V is bounded as $\lambda_{\min}(P)|\tilde{z}_1 - \tilde{z}_2|^2 \leq V(\tilde{z}_1, \tilde{z}_2) \leq \lambda_{\max}(P)|\tilde{z}_1 - \tilde{z}_2|^2$. During flows, namely $(\tilde{z}_1, \tilde{z}_2) \in \widetilde{C}$, it follows that

$$\begin{aligned} &\langle \nabla_{\tilde{z}_1} V(\tilde{z}_1), f(\tilde{z}_1)\rangle + \langle \nabla_{\tilde{z}_2} V(\tilde{z}_2), f(\tilde{z}_2)\rangle \\ &\qquad = (\tilde{z}_1 - \tilde{z}_2)^\top (PA + A^\top P)(\tilde{z}_1 - \tilde{z}_2). \end{aligned} \tag{11}$$

When the matrix A is Hurwitz, we know that P can be chosen such that $PA + A^\top P < 0$ and (10b) is satisfied. During jumps, for the case $\tilde{z}_1, \tilde{z}_2 \in D$, we obtain

$$V(g(\tilde{z}_1), g(\tilde{z}_2)) - V(\tilde{z}_1, \tilde{z}_2) = (\tilde{z}_1 - \tilde{z}_2)^\top (B^\top PB - P)(\tilde{z}_1 - \tilde{z}_2).$$

When the matrix B is such that $|B| \leq 1$, there exists P such that $B^\top PB - P \leq 0$ and (10d) is satisfied. Lastly, consider the case $\tilde{z}_1 \in D$ and $\tilde{z}_2 \in C \setminus D$, it follows that $V(g(\tilde{z}_1), \tilde{z}_2) - V(\tilde{z}_1, \tilde{z}_2) = \tilde{z}_1^\top (B^\top PB - P)\tilde{z}_1 - 2\tilde{z}_1^\top (B^\top P - P)\tilde{z}_2$, which, under the assumptions, is less than or equal to zero. Hence, (10c) holds, the set $\widetilde{\mathcal{A}}$ is UGAS for $\widetilde{\mathcal{H}}$, and by Proposition 4.4, $\mathcal{H}$ is δUGAS. To show that the required conditions are restrictive, take $f(z) = -0.1z$ and $g(z) = -0.5z$ with flow set $C = [0, \infty)$ and jump set $D = (-\infty, 0)$ and note that $P = 4$ gives $PA + A^\top P = -0.8 < 0$, $|B| = 0.5 < 1$, $B^\top PB - P = -3 < 0$ and that for each $\eta_1 \in D$, $\eta_2 \in C \setminus D$, $-\eta_1^\top (B^\top P - P)\eta_2 \leq 0$ holds. On the other hand, the solutions to $\mathcal{H}$ do not have more than one jump. $\triangle$

VII. Conclusion

In this paper, we introduced and studied several notions of incremental stability for hybrid systems. Incremental stability involves a convergence property where solutions converge to each other. To study incremental stability for hybrid systems, an auxiliary system was proposed. Sufficient and necessary conditions for a hybrid system to be incrementally stable and incrementally partially stable were provided and illustrated in examples.

References

[1] D. Angeli. A Lyapunov approach to incremental stability properties. *IEEE Transactions on Automatic Control*, 47(3):410–421, Mar 2002.

[2] Q. Pham, N. Tabareau, and J. J. Slotine. A contraction theory approach to stochastic incremental stability. *IEEE Transactions on Automatic Control*, 54(4):816–820, April 2009.

[3] B. S. Ruffer, N. van de Wouw, and M. Mueller. Convergent systems vs. incremental stability. *Systems & Control Letters*, 62(3):277–285, 2013.

[4] R. Goebel, R. G. Sanfelice, and A. R. Teel. *Hybrid Dynamical Systems: Modeling, Stability, and Robustness*. Princeton University Press, New Jersey, 2012.

[5] R. Goebel, R. G. Sanfelice, and A. R. Teel. Hybrid dynamical systems. *IEEE Control Systems Magazine*, 29(2):28–93, April 2009.

[6] R. G. Sanfelice, R. Goebel, and A. R. Teel. Invariance principles for hybrid systems with connections to detectability and asymptotic stability. *IEEE Transactions on Automatic Control*, 52(12):2282–2297, 2007.

[1] See [4, Definition 3.16].