A Hybrid PID Design for Asymptotic Stabilization with Intermittent Measurements

Daniel Lavell, Sean Phillips, and Ricardo G. Sanfelice

Abstract— **In this paper, we propose a modeling and design technique for a proportional-integral-derivative (PID) controller in the presence of aperiodic intermittent sensor measurements. Using classical control design methods, PID controllers can be designed when measurements are available periodically, at discrete time instances, or continuously. Unfortunately, such design do not apply when measurements are available intermittently. Using the hybrid inclusions framework, we model the continuous-time plant to control, the mechanism triggering intermittent measurements, and a hybrid PID control law defining a hybrid closed-loop system. We provide sufficient conditions for uniform global asymptotic stability using Lyapunov set stability methods. These sufficient conditions are used for the design of the gains of the hybrid PID controller. Also, we propose relaxed sufficient conditions to provide a computationally tractable design method leveraging a polytopic embedding approach. The results are illustrated via numerical examples.**

I. Introduction

Proportional-integral-derivative controllers are incredibly popular in engineering applications; see, e.g., [1], [2], [3]. For continuous-time systems, a PID control law is given by

$$u(t) = K_P e(t) + K_I \int_0^t e(s)ds + K_D \dot{e}(t), \qquad (1)$$

where $t \geq 0$, u is the input to the system being controlled (the plant), e is the error between the state and the reference to be tracked, and K_P, K_I, and K_D are the proportional, integral, and derivative parameters (or gains) to be designed, respectively. Several design techniques are available to determine the three parameters in the PID controller to meet design specifications such as rise time, settling time, and overshoot [2], [4]. However, classical design methods require continuous or periodically sampled measurements of the output, which may not be practical in certain applications [2], [3]. Namely, when the measurements are available only at aperiodic, intermittent time instances novel methods for the design of the control law in (1) are needed, and unavoidably, demand the use of hybrid systems tools.

Some design techniques for PID controllers that could have potential for the settling of intermittent, aperiodic sampling are available in the literature. A multi-rate PID control law is considered in [5] through discretizing the continuous-time dynamics and considering a delayed sensor to input signal dependent on the sampling rate. It should be pointed out that first-order reset elements have shown to be advantageous towards the performance of PID controllers [6]. On the other hand, with the popularization of systems that contain both continuous and discrete dynamics, there are several novel approaches with the potential for the design of PID controllers under intermittence. In [7], the authors consider a continuous-time system and design an event-triggered control law using Lyapunov-based analysis. In [8], the authors utilize an impulsive systems approach to design a static feedback controller for a continuous-time linear time-invariant system and uses an estimate event-based trigger to update the controller. Hybrid controllers with sporadic measurements have been studied in [9], [10] but to address different problems. The authors of [9] consider the problem of observer design under sporadic measurements. In [10], the design of a hybrid feedback controller for consensus intermittent communication over a network of agents is proposed.

In this paper, we consider the case when the plant is a linear time-invariant system, but the output is only measured at, potentially non-periodic, isolated time instances. Namely, subsequent measurements can occur any time within a known bounded window of ordinary time. To cope with intermittency, we introduce a hybrid PID control law akin to the continuous-time one in (1), that allows for continuous evolution of the state as well as impulsive measurements and control updates. Due to the continuous-time and impulsive dynamics of the closed-loop system we utilize the hybrid systems framework in [11] for modeling, analysis, and design. Using Lyapunov-based tools for uniform global asymptotic stability of compact sets, we provide sufficient conditions on the parameters of the hybrid PID controller to guarantee such stability property. Though these conditions are nonlinear and must be solved at infinitely many points, a polytopic embedding approach is shown to yield a computationally tractable design method to determine the parameters of the hybrid PID controller. Numerical simulations validate these results.

The paper is organized as follows. Section II provides basic background. Section III presents the system under consideration and provides a motivational example. Section IV models the closed-loop system as a hybrid system and gives examples for the special cases of proportional, proportional-integral, and proportion-derivative control laws. Section V gives the main results and design methods. Section VI illustrates the main results and design through via examples.

D. Lavell and R. G. Sanfelice are with the Electrical and Computer Engineering Department, University of California at Santa Cruz, {dlavell, ricardo}@ucsc.edu. Their research has been partially supported by the National Science Foundation under CAREER Grant no. ECS-1450484 and Grant no. CNS-1544396, the Air Force Office of Scientific Research under Grant no. FA9550-16-1-0015, CITRIS, and the Banatao Institute at the University of California.

S. Phillips is with the Air Force Research Laboratory, Space Vehicles Directorate, Kirtland AFB, NM, 87117.

II. Notation and Preliminaries

A. Notation

We denote P being positive definite as $P > 0$ and being negative definite as $P < 0$. Given $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$, the pair (x, y) is equivalent to $[x^\top, y^\top]^\top$. The distance from a vector $x \in \mathbb{R}^n$ to a closed set $\mathcal{A} \subset \mathbb{R}^n$ is $|x|_{\mathcal{A}} := \inf_{y \in \mathcal{A}} |x - y|$. A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a class-$\mathcal{K}$ function, also written $\alpha \in \mathcal{K}$, if α is zero at zero, continuous, strictly increasing; it is said to belong to class-$\mathcal{K}_\infty$, also written $\alpha \in \mathcal{K}_\infty$, if $\alpha \in \mathcal{K}$ and is unbounded; α is positive definite, also written $\alpha \in \mathcal{PD}$, if $\alpha(s) > 0$ for all $s > 0$ and $\alpha(0) = 0$. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a class-$\mathcal{KL}$ function, also written $\beta \in \mathcal{KL}$, if it is nondecreasing in its first argument, nonincreasing in its second argument, $\lim_{r \to 0^+} \beta(r, s) = 0$ for each $s \in \mathbb{R}_{\geq 0}$, and $\lim_{s \to \infty} \beta(r, s) = 0$ for each $r \in \mathbb{R}_{\geq 0}$. Given a function f, its domain is denoted by $\operatorname{dom} f$. Given a set X, $\operatorname{co}X$ represents the convex hull of X.

B. Hybrid Systems

This section introduces the main notions and definitions on hybrid systems used throughout this paper. More information on such systems can be found in [11]. For the purposes of this paper, a hybrid system $\mathcal{H}$ is given in the compact form

$$\mathcal{H} : \left\{ \begin{array}{lcll} \dot{x} & = & f(x) & x \in C, \\ x^+ & \in & G(x) & x \in D, \end{array} \right. \tag{2}$$

where $x \in \mathbb{R}^n$ is the state and the data of the hybrid system, denoted (C, f, D, G), is defined as follows:

- $f : \mathbb{R}^n \to \mathbb{R}^n$ is a single-valued map defining the flow map capturing the continuous dynamics;
- $C \subset \mathbb{R}^n$ defines the flow set on which f is effective;
- $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a set-valued map defining the jump map and models the discrete behavior;
- $D \subset \mathbb{R}^n$ defines the jump set, which is the set of points from where jumps are allowed.

Solutions ϕ to $\mathcal{H}$ are parameterized by (t, j), where $t \in \mathbb{R}_{\geq 0} := [0, \infty)$ counts ordinary time and $j \in \mathbb{N} := \{0, 1, 2, \dots\}$ counts the number of jumps. The domain $\operatorname{dom} \phi \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ is a hybrid time domain if for every $(T, J) \in \operatorname{dom} \phi$, the set $\operatorname{dom} \phi \cap ([0, T] \times \{0, 1, \dots, J\})$ can be written as the union of sets $\cup_{j=0}^{J} (I_j \times \{j\})$, where $I_j := [t_j, t_{j+1}]$ for a time sequence $0 = t_0 \leq t_1 \leq t_2 \leq \cdots \leq t_{J+1}$. The t_j's with $j > 0$ define the time instants when the state of the hybrid system jumps and j counts the number of jumps. A solution to $\mathcal{H}$ is called maximal if it cannot be extended; i.e., it is not a truncated version of another solution. It is called complete if its domain is unbounded. A solution is Zeno if it is complete and its domain is bounded in the t direction. A solution is precompact if it is complete and bounded.

Definition 2.1: (uniform global asymptotic stability) Let a hybrid system $\mathcal{H}$ be defined on $\mathbb{R}^n$ and $\mathcal{A} \subset \mathbb{R}^n$ be closed. The set $\mathcal{A}$ is said to be

- uniformly globally stable (UGS) for $\mathcal{H}$ if there exists $\alpha \in \mathcal{K}_\infty$ such that any solution ϕ to $\mathcal{H}$ satisfies $|\phi(t, j)|_{\mathcal{A}} \leq \alpha(|\phi(0, 0)|_{\mathcal{A}})$ for all $(t, j) \in \operatorname{dom} \phi$;
- uniformly globally attractive (UGA) for $\mathcal{H}$ if for each $\varepsilon > 0$ and $r > 0$ there exists $T > 0$ such that every maximal solution ϕ to $\mathcal{H}$ is complete and if $|\phi(0, 0)|_{\mathcal{A}} \leq r$, $(t, j) \in \operatorname{dom} \phi$ and $t + j \geq T$ then $|\phi(t, j)|_{\mathcal{A}} \leq \varepsilon$;
- uniformly globally asymptotically stable (UGAS) for $\mathcal{H}$ if it is both UGS and UGA.

Sufficient conditions for UGAS of a closed set $\mathcal{A}$ can be found utilizing a Lyapunov function candidate; for more information regarding such properties, see [11].

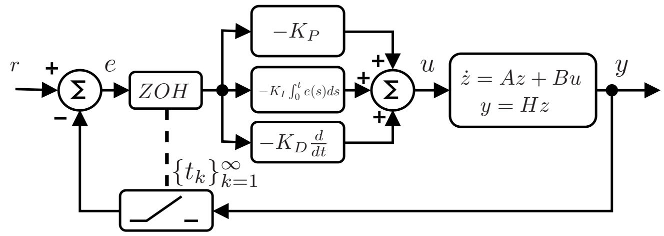

Fig. 1. Block diagram of the closed loop system with each controller and measurement timing.

III. Motivational Example and Problem Statement

Consider a continuous linear time-invariant system defining the plant, with state $z \in \mathbb{R}^n$ and input $u \in \mathbb{R}^m$ given by

$$\dot{z} = Az + Bu \tag{3}$$

where A and B are matrices of appropriate dimension. We consider the case when the output of the plant

$$y = Hz \in \mathbb{R}^p \tag{4}$$

is available for the purposes of control at isolated time instances. More precisely, the output y is available to the controller when $t \in \{t_k\}_{k=1}^{\infty}$, where the sequence of times $\{t_k\}_{k=1}^{\infty}$ satisfies

$$T_1 \leq t_{k+1} - t_k \leq T_2 \quad \forall k \in \mathbb{N} \setminus \{0\}, \qquad 0 \leq t_1 \leq T_2 \tag{5}$$

with T_1 and T_2 such that $0 < T_1 \leq T_2$. The parameter T_1 denotes the minimum time for samples while T_2 denotes the maximum time in between samples, which is known in the literature as the maximum allowable transfer interval (MATI); see, e.g., [12]. Figure 1 depicts a feedback closed-loop system using a PID controller where the output is available at times given by the sequence of times $\{t_k\}_{k=1}^{\infty}$ as indicated by the switch therein. Note that the closed-loop system includes a reference signal r to be tracked.

To illustrate the effects of intermittent measurements of the output on a PID feedback loop, consider a mass-spring system where only position can be measured. The state $z = (z_1, z_2) \in \mathbb{R} \times \mathbb{R}$, where z_1 is position and z_2 is velocity of the mass, respectively. Namely, the system in (3)-(4) is defined by matrices

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad H = \begin{bmatrix} 1 & 0 \end{bmatrix} \tag{6}$$

Suppose the goal is to design a PID controller to meet the following specifications: rise time $t_r \leq 0.2$ seconds, settling

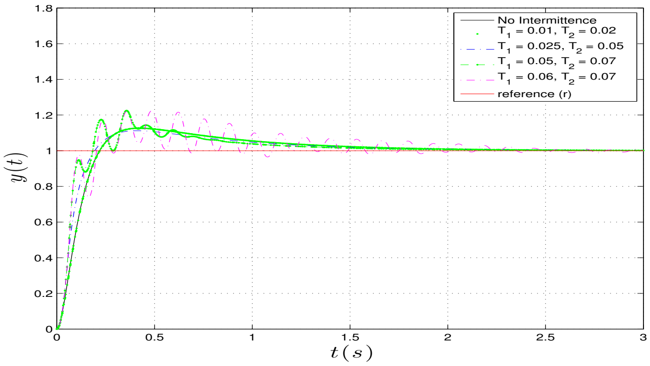

Fig. 2. Trajectories of a mass-spring system tracking a reference signal $r = 1$ (red). The black trajectory was generated when the closed-loop system had continuous sensor measurements. The remaining trajectories illustrate the degradation of the closed-loop system when sensor measurements arrive sporadically.

time $t_s \leq 2$ seconds, and overshoot $M_p \leq 15\%$. When the output y is available continuously, the gains $K_P = 250$, $K_I = 350$, and $K_D = 30$ generate a closed-loop system satisfying the given specifications. The output trajectory in black in Figure 2 shows the response of the system with such feedback. Unfortunately, when the same feedback gains in the PID controller are used with the output measured intermittently, at times satisfying (5), leads to degradation of performance. Figure 2 shows the output trajectories to the system for the above PID gains with a sample-and-hold feedback scheme for increasing values of T_1 and T_2. Note that, even for small parameters $T_1 = 0.06$ seconds and $T_2 = 0.07$ seconds (shown in magenta in Figure 2), the overshoot increases by 50% compared to the continuous feedback case. Also, for such choices, the settling time is well beyond specification as oscillations are still present beyond 3 seconds. If T_1 and T_2 are large enough, then there is no guarantee that convergence will happen at all.

IV. The Hybrid PID

In this section, we present a modeling approach of the PID controller in (1) when the measurements occur at times given by (5). Due to the continuous dynamics of the plant in (3), the intermittent sensor measurements communicating at times given by (5), and the control law in (1) (yet to be designed), the system naturally has both continuous and discrete dynamics. Therefore, we model the closed-loop systems using the hybrid systems framework presented in [11]. In this paper, for simplicity, we consider the case when the reference signal is zero, but the results and ideas can be extended to the case when the reference is generated by an exosystem; e.g., as in [13].

A. Intermittent Measurement Model

The output of the plant is measured at impulsive times satisfying (5). To generate all possible such sequences, we define a timer state, denoted by $\tau \in [0, T_2]$, which decreases continuously in ordinary time t and, when it reaches zero, it is reset to a point in the interval $[T_1, T_2]$. The timer can be modeled as an autonomous hybrid inclusion given by

$$\begin{cases} \dot{\tau} \ = -1 & \tau \in [0, T_2] \\ \tau^+ \in [T_1, T_2] & \tau = 0 \end{cases} \tag{7}$$

Such a timer defines a hybrid system with solutions having jump times t_j satisfing (5); for more details on the use of such timers, see [9], [10].

B. Hybrid PID Model

Next, we introduce each component of the proposed hybrid PID controller. The hybrid PID controller has three components: the proportional component, v_P; the integral component, v_I; and the derivative component, v_D. With a slight abuse of notation, we denote the output of the controller by the state u, which evolves according to zero-order hold dynamics. Namely, during the intervals of time between successive measurement updates, u is held constant, and, when the controller receives a new measurement, we update it with the components of the controller. The hybrid PID controller is given by

$$\begin{cases} \dot{u} \ = 0 & \tau \in [0, T_2] \\ u^+ = v_P + v_I + v_D & \tau = 0 \end{cases} \tag{8}$$

where v_P, v_I, and v_D are defined explicitly below.

1) Proportional Action: Following the construction in (1), the contribution of the proportional component v_P of the hybrid PID is proportional to the measurement received. It follows that, at jumps, the component v_P is given by $v_P = -K_P y = -K_P H z$.

2) Integral Action: In classical state-space control design, an integral controller requires the introduction of an auxiliary state which 'memorizes' the integral of the error between the state and reference [2], [3]. To capture such a mechanism in the case of intermittent measurements, we introduce two states: a memory state m_s and an integral state z_I. We use $z_I \in \mathbb{R}^p$ as the state storing an approximation of the running total integral. The memory state $m_s \in \mathbb{R}^p$ is used to store the most recent measurement of the output y. The memory state is updated when a new output measurement is available, which according to (7) is when the timer τ is equal to zero. Between sensor measurements, the integral state z_I evolves according to $\dot{z}_I = m_s$ while the memory state m_s remains constant. Then, the integral control law is then implemented as

$$v_I = -K_I z_I. \tag{9}$$

3) Derivative Action: To implement the derivative action v_D, first, consider the case when only the derivative term in (1) is present. Therefore we have that

$$v_D = -K_D \dot{y} = -K_D H(Az + Bv_D) \tag{10}$$

which leads to $v_D = -(I + K_D HB)^{-1} K_D HAz$ where, implicitly, we assume that $I + K_D HB$ is invertible. Combining the proportional and integral controller, we have

$$v_D = -(I + K_D HB)^{-1} K_D H(Az - BK_P Hz - BK_I z_I). \tag{11}$$

C. Hybrid Closed-loop System

To write the resulting hybrid closed-loop system combining the three control actions in Sections IV-B.1, IV-B.2, and IV-B.3, we define the state of the hybrid system $\mathcal{H}$ as $x = (x_1, x_2) \in \mathcal{X} := \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^m \times \mathbb{R}^p \times [0, T_2]$, where $x_1 = (z, z_I, u, m_s)$ and $x_2 = \tau$. The resulting closed-loop system with the plant in (3), PID controller in (8), and timer in (7) has data (C, f, D, G) given by

$$\begin{aligned} f(x) &:= \begin{bmatrix} A_f x_1 \\ -1 \end{bmatrix} \qquad \forall x \in C := \mathcal{X} \\ G(x) &:= \begin{bmatrix} A_g x_1 \\ [T_1, T_2] \end{bmatrix} \qquad \forall x \in D := \{x \in \mathcal{X} : \tau = 0\} \end{aligned} \tag{12}$$

The matrices A_f and A_g are given by

$$A_f = \begin{bmatrix} A & 0 & B & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, A_g = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ -\tilde{K}_P - \tilde{K}_D & -\tilde{K}_I & 0 & 0 \\ H & 0 & 0 & 0 \end{bmatrix} \tag{13}$$

where

$$\begin{aligned} \tilde{K}_P &= K_P H - (I + K_D H B)^{-1} K_D H B K_P H \\ \tilde{K}_I &= K_I - (I + K_D H B)^{-1} K_D H B K_I \\ \tilde{K}_D &= (I + K_D H B)^{-1} K_D H A \end{aligned} \tag{14}$$

Note that the definitions of $\tilde{K}_P$, $\tilde{K}_I$, and $\tilde{K}_D$ depend on (K_P, K_D), (K_I, K_D), and K_D respectively[1]. We will treat $\tilde{K}_P$, $\tilde{K}_I$, and $\tilde{K}_D$ as our design parameters[2].

Remark 4.1: If the parameters $\tilde{K}_P$, $\tilde{K}_I$, and $\tilde{K}_D$ are known, then the values of K_P, K_D and K_I can be recovered. Namely, the parameter K_D can be solved for directly as long as the invertibility condition on $(I + K_D H B)$ holds. In that case K_D can be used to solve for the parameters K_P and K_I directly. For example, consider the case when the dynamics and control inputs are scalars, then if follows that the control parameters K_P, K_I (in terms of K_D), and K_D are given as $K_P = \tilde{K}_P(1 + K_D H B)/H$, $K_I = \tilde{K}_I(1 + K_D H B)$, and $K_D = \tilde{K}_D/(HA - \tilde{K}_D H B)$, respectively.

Given the hybrid closed-loop system in (12), and parameters $0 < T_1 \leq T_2$, our goal is to design the parameters $\tilde{K}_P$, $\tilde{K}_I$, and $\tilde{K}_D$ of the hybrid PID controller such that the compact set

$$\mathcal{A} = \{(z, z_I, u, m_s, \tau) \in \mathcal{X} : z = z_I = u = m_s = 0\} \tag{15}$$

is uniformly globally asymptotically stable. Note that this set captures the usual equilibrium point, namely, the origin, to which (3) is stabilized to via PID control when the reference is zero.

D. Special Cases

Next, we showcase three special cases of the hybrid PID controller in (8) that not only simplify its construction but also find wide use in applications.

[1]For the scalar case, the expressions for $\tilde{K}_P$ and $\tilde{K}_I$ are so that K_P and K_I can be chosen to yield desired values of $\tilde{K}_P$ and $\tilde{K}_I$, even though K_D plays a role in their definition.

[2]The inveritibility condition on $(I + K_D H B)$ is only necessary if the gains K_P, K_I, and K_D need to be derived specifically.

1) Proportional Control Case: In the case when the control law implements the proportional action only, the states z_I and m_s in (12) can be removed. In this particular case, the state of the closed-loop system is $x = (x_1, x_2)$ with $x_1 = (z, u)$ and $x_2 = \tau$. The flow map, flow set, jump map, and jump set are still given as in (12) but with obvious changes on dimensions. The matrices in (13) reduce to

$$A_f = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}, \quad A_g = \begin{bmatrix} I & 0 \\ -\tilde{K}_P & 0 \end{bmatrix} \tag{16}$$

with $\tilde{K}_P = K_P H$. In this case, the desired set to stabilize is

$$\mathcal{A} = \{(z, u, \tau) \in \mathbb{R}^n \times \mathbb{R}^m \times [0, T_2] : z = u = 0\} \tag{17}$$

2) Proportional-Integral Control Case: The model in (12) for only proportional-integral (PI) control still requires the memory states m_s and z_I used to approximate integration between sampling events. The state of the closed-loop system is $x = (x_1, x_2)$ with $x_1 = (z, z_I, u, m_s)$ and $x_2 = \tau$. Definitions of A_f and A_g follow directly from (13) with the derivative gain $K_D = 0$, resulting in

$$A_f = \begin{bmatrix} A & 0 & B & 0 \\ 0 & 0 & 0 & I \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, A_g = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ -\tilde{K}_P & -\tilde{K}_I & 0 & 0 \\ H & 0 & 0 & 0 \end{bmatrix} \tag{18}$$

with $\tilde{K}_P = K_P H$ and $\tilde{K}_I = K_I$. The flow map, flow set, jump map, and jump set are still given as in (12) and the set to stabilize remains as in (15).

3) Proportional-Derivative Control Case: In the case of proportional-derivative (PD) control only, the components of the state $x = (x_1, x_2)$ in the model (12) simplify to $x_1 = (z, u)$ and $x_2 = \tau$, as the integration states z_I and m_s are no longer needed. The matrices A_f and A_g reduce to

$$A_f = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}, \qquad A_g = \begin{bmatrix} I & 0 \\ -\tilde{K}_P - \tilde{K}_D & 0 \end{bmatrix} \tag{19}$$

where the gains $\tilde{K}_P$ and $\tilde{K}_D$ are defined in (14), while the definition of the data of (12) remains the same, modulo obvious changes of dimensions. The set to stabilize is given in (17). In the next section, we provide tools for the design of the gains in (14).

V. Design Conditions for the Hybrid PID

The following result gives sufficient conditions for uniform global asymptotic stability of the set $\mathcal{A}$ in (15) for the hybrid system in (12) in terms of matrix inequalities. The result holds for the generic matrices A_f and A_g in (13), and covers the special cases in Sections IV-D.1, IV-D.2, and IV-D.3.

Following [9] and [11], we establish uniform global asymptotic stability of the set $\mathcal{A}$ in (15) using a Lyapunov-based analysis following the ideas in [11, Example 3.14]. To that end, we consider the following Lyapunov function candidate

$$V(x) = W(\exp(A_f \tau) x_1) \qquad \forall x \in \mathcal{X} \tag{20}$$

where $W(s) = s^\top Ps$ with P a symmetric positive definite matrix. Note that (20) is a Lyapunov function candidate according to Definition 3.16 in [11], in particular, V is continuously differentiable everywhere. We have the following result.

Theorem 5.1: Let T_1 and T_2 be positive scalars such that $T_1 \leq T_2$. Suppose there exist matrices $\tilde{K}_P$, $\tilde{K}_I$, and $\tilde{K}_D$, and a positive definite symmetric matrix P satisfying

$$\Gamma(\nu)^\top P\Gamma(\nu) - P < 0 \qquad \forall \nu \in [T_1, T_2] \tag{21}$$

where $\Gamma(\nu) = \exp(A_f\nu)A_g$, and the matrices A_f and A_g are given in (13)*. Then, the set $\mathcal{A}$ in* (15) *is uniformly globally asymptotically stable for the hybrid system $\mathcal{H}$ with data as in* (12)*.*

Proof Sketch: Consider the Lyapunov function given in (20) with $P = P^\top > 0$. First note that there exists $0 < \underline{c} < \overline{c}$ such that, for each $x \in C \cup D \cup G(D)$, V satisfies $\underline{c}|x|^2_{\mathcal{A}} \leq V(x) \leq \overline{c}|x|^2_{\mathcal{A}}$. During flows, namely, for each $x \in C$, there is no change in V. It follows that at jumps through the continuity in V of (21) there exists $\epsilon > 0$ such that the change in V is given by $V(g) - V(x) \leq -\epsilon|x|^2_{\mathcal{A}}$ for each $x \in D, g \in G(x)$. From Proposition 6.10 in [11], every maximal solution ϕ to $\mathcal{H}$ is complete. Moreover, the intervals of flow time between jumps for each maximal solution ϕ is bounded as $t \leq (j+1)T_2$ for all $(t,j) \in \operatorname{dom}\phi$, where the given T_2 is positive. Moreover, for every $(t,j) \in \operatorname{dom}\phi$ such that $t + j \geq T$ it follows that $j \geq \frac{T}{T_2+1} - \frac{T_2}{T_2+1}$ from Theorem 3.24 in [11], the set $\mathcal{A}$ in (15) is uniformly globally asymptotically stable for $\mathcal{H}$ in (12). $\square$

Remark 5.2: Due to the nonlinearities, solving (21) for P, $\tilde{K}_P$, $\tilde{K}_I$, and $\tilde{K}_D$ may not be numerically tractable. When the update times are periodic, namely, when $T_1 = T_2$ and the controller gains are given a priori, we can use a convex optimization solver like CVX in [14] to solve for P. However, when the gains are being designed, (21) contains nonlinear terms and must be evaluated over infinitely many points $\nu \in [T_1, T_2]$.

To alleviate the issues pointed out in Remark 5.2, we provide a systematic and numerically tractable approach using the polytopic embedding in [9] to solve for the controller gains $\tilde{K}_D$, $\tilde{K}_I$, $\tilde{K}_P$, and the matrix P. First, following Proposition 1 in [9], we use the Projection Lemma and Schur's complement to get an equivalent form for the inequality in (21).

Theorem 5.3: Let T_1 and T_2 be positive scalars such that $T_1 \leq T_2$. Given the matrices A, B, and H defining the plant dynamics and output, the matrices A_f and A_g in (13)*, and the matrix P satisfy* (21) *if and only if there exists a matrix $F \in \mathbb{R}^{n\times n}$ satisfying*

$$\begin{bmatrix} -(F+F^\top) & FA_g & \exp(A_f^\top\nu)P \\ * & -P & 0 \\ * & * & -P \end{bmatrix} < 0 \quad \forall \nu \in [T_1, T_2] \tag{22}$$

A similar construction to Theorem 5.3 is proposed in [9] for the design of a hybrid observer when measurements are available intermittently.

Theorem 5.3 gives an equivalent form of (21) that is linear with respect to P, F, and A_g. However, this condition still needs to be checked for infinitely many values of $\nu \in [T_1, T_2]$. One method to deal with the dense set $[T_1, T_2]$ is to embed $\exp(A_f\nu)$ into finitely many polytopes; that is, find matrices $\{X_1, X_2, \ldots, X_w\}$ such that $\exp(A_f\nu) \in \operatorname{co}\{X_1, X_2, \ldots, X_w\}$ for each $\nu \in [T_1, T_2]$.

Corollary 5.4: Let T_1 and T_2 be positive scalars such that $T_1 \leq T_2$. Let the matrices $\{X_1, X_2, \ldots, X_w\}$ satisfy

$$\exp(A_f[T_1, T_2]) \subset co\{X_1, X_2, \ldots, X_w\}.$$

If there exist matrices J and F, and a positive definite symmetric matrix P such that, for each $i \in \{1, 2, \ldots, w\}$,

$$\begin{bmatrix} -(F+F^\top) & J & X_iP \\ \star & -P & 0 \\ \star & \star & -P \end{bmatrix} < 0 \tag{23}$$

where the entries F_{ik} of F satisfy[3]

$$\begin{bmatrix} F_{11} - F_{13}K_{PD} + F_{14}H & F_{12} - F_{13}\tilde{K}_I \\ F_{21} - F_{23}K_{PD} + F_{24}H & F_{22} - F_{23}\tilde{K}_I \\ F_{31} - F_{33}K_{PD} + F_{34}H & F_{32} - F_{33}\tilde{K}_I \\ F_{41} - F_{43}K_{PD} + F_{44}H & F_{42} - F_{43}\tilde{K}_I \end{bmatrix} = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \\ J_{31} & J_{32} \\ J_{41} & J_{42} \end{bmatrix} \tag{24}$$

with $K_{PD} = \tilde{K}_P + \tilde{K}_D$ and

$$J = \begin{bmatrix} J_{11} & J_{12} & 0 & 0 \\ J_{21} & J_{22} & 0 & 0 \\ J_{31} & J_{32} & 0 & 0 \\ J_{41} & J_{42} & 0 & 0 \end{bmatrix}, \tag{25}$$

then the matrices P and $FA_g = J$ satisfy condition (21)*.*

VI. EXAMPLES

We illustrate the design of a hybrid PID in examples. Simulations use the HyEQ Toolbox in Matlab[4] [15].

Example 6.1: In this example we illustrate Theorem 5.1. Consider the mass-spring system with matrices as in (6). Let $\tilde{K}_P = 10$, $\tilde{K}_I = 4$, and $\tilde{K}_D = 4$. The time bounds T_1 and T_2 are chosen as $T_1 = 0.1$ and $T_2 = 0.25$. Using CVX [14] and defining matrices A_f and A_g as in (13), we can solve for P while enforcing the condition in (21) and that $P = P^\top > 0$. Components of a solution to the closed-loop system and the value of V along it are shown in Figure 3 (projected to the t axis). Trajectories for the case of continuous measurements and same parameters are also shown. Under intermittent output measurements, we are able to guarantee uniform global asymptotic stability of the desired set. Figure 3 also shows the control input to the system over time. Note that the value of the control signal u is held constant between output measurements – these events

[3]Note that there are multiple options for constraining F and J according to (24). For instance, when conditions $F_{23} = F_{33} = F_{43} = 0$ and $F_{13} = I$ are imposed, then $\tilde{K}_{DP} = J_{11} - F_{11} - F_{14}C$ and $\tilde{K}_I = J_{12} - F_{12}$.

[4]The MATLAB code for simulations presented in this paper are available at GitHub repository https://github.com/HybridSystemsLab/HybridPID.git

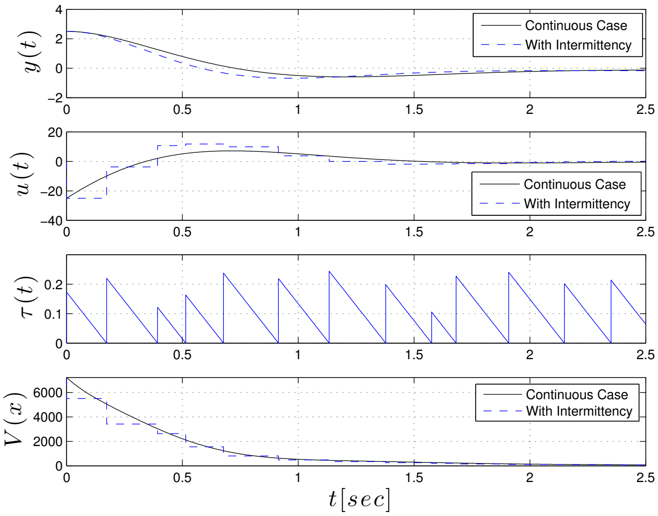

Fig. 3. The position of the mass from a mass-spring-damper system when using a PID controller and for a given reference signal $r = 0$ is shown in the first plot. The control signal generated from the PID controller with a sample and hold mechanism in the feedback loop is depicted in the second graph. The timer triggering the sporadic events is seen in the third plot. The Lyapunov function along a solution is shown in the bottom plot.

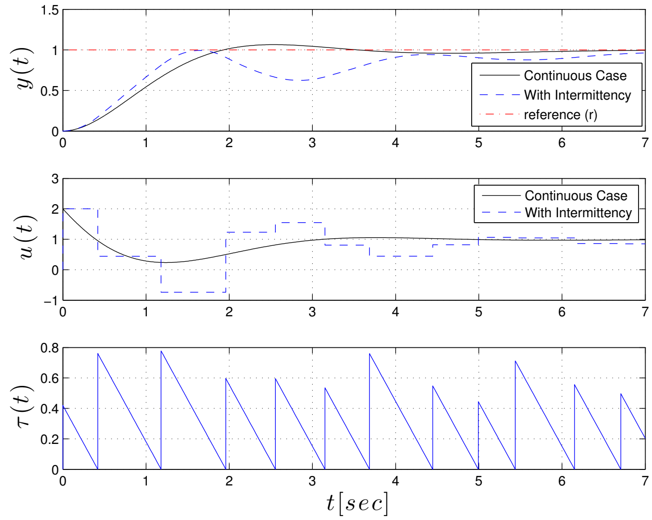

Fig. 4. The closed-loop state response for PI controller given constant reference signal $r = 1$ in the top plot shows solutions for the mass-spring system when outputs are measured continuously (black) and sporadically (blue). The middle plot illustrates the associated control signals generated by the PI controller in both cases. The bottom plot shows values of the timer τ used to trigger the sporadic output measurements.

are not periodic. Simulation results validate Theorem 5.1 as confirmed by the evolution of the Lyapunov function V at the bottom of Figure 3.

Example 6.2: Consider the design of a PI controller as in Section IV-D.2 for the mass-spring system with matrices in (6), but now with the ability to observe both position z_1 and velocity z_2. Given a constant reference signal, a PI controller should have the steady state error $e_{ss} = 0$. We design appropriate values of $\tilde{K}_P$ and $\tilde{K}_I$ to show that with sporadic output measurements triggered at times satisfying (5), the steady state error of the closed-loop system with PI control is zero. To this end, pick $\tilde{K}_P = 2$ and $\tilde{K}_I = 1$, and define A_f and A_g as in (18). Figure 4 compares the state response and associated input signal for the hybrid closed-loop system given a unit step input $r = 1$. The initial state of the plant is zero and the time bounds are chosen as $T_1 = 0.4$ and $T_2 = 0.8$.

VII. Conclusion

We have shown a systematic approach to designing a PID controller where the measurements occur at intermittent instances. By modeling the closed-loop system using the hybrid inclusion framework, we provided sufficient conditions for uniform global asymptotic stability for the set of interest and give a detailed approach for design following a polytopic embedding approach. Future work for this research is to investigate dynamic gain scheduling to maximize convergence rate while minimizing overshoot.

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